2025, volume 2 , Issues 5 PSIJK 55-61

Pure Sciences International Journal of Kerbala
Journal Homepage:https://journals.uokerbala.edu.iq/index.php/psijk

On Pure Ideals: A Review

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P A P E R I N F O

Received: 11 November 2024
Accepted: 2 February 2025
Published: 31 March 2025

Keywords:
Pure ideals, Strongly Pure ideals, PIP-ring, PIGP-ring, MRGP-ring, Commutative ring.

A B S T R A C T

Ring theory is one of the branches of an abstract algebra. This field is the study of a mathematical system with two binary operations. In this branch, many articles have studied this algebraic structure and presented some new works. However, the concept of purity has been studied before more than 40 years ago, especially the relation between the pure ideal and some other ideals on the given ring. In this paper, we survey the important results that concern with pure ideals. Some different types of ideals have been discussed such as N-pure ideals, z-ideals, Π-pure ideals and strongly pure ideals. Moreover, some recent results based on the work of several researchers have been summarized. On the other hand, regarding these types of ideals, some questions have been presented. Furthermore, many important results about various types of rings which are based on the notion of pure ideals have been studied.

1. INTRODUCTION

An abstract algebra is one of the major branches in Mathematics. The name of algebra is mentioned in the book of the mathematician Muhammad ibn Musa al-Khwarizmi. An abstract algebra is a broader field, and this field is concerned with the study of algebraic structures and the relations between them. A number of articles have been reviewed in ring theory and others in group theory. In other words, some authors have been published their articles which concerned with specific topic in rings or groups. For instance, in 2005 Nicholson and Zhou in [1] presented such study in order to summarize the developments on the concept of clean rings. In their study, many results are presented for exchange ring (changing property), group rings, semi-boolean ring and the connection among C*-algebras and clean rings of continuous functions. In addition, they presented some results which concentrated on the strongly clean rings. Regarding these topics, they stated some questions, some of which have been answered and others are not. Furthermore, another study with different direction has been determined in [2]. This study was a survey on the strongly clean rings. Moreover, a brief survey paper on the rings generated by units has been presented in [3]. On the other hand, a short survey that emphasizes the theorems of Schur and Baer has been written in [4]. This study discussed some of the recent results that generalize these theorems. In this paper, we present a short survey on the notion of pure ideals with related concepts and discuss them in detail. The present paper is an attempt to provide some results that concerned pure ideals and their related concepts in a suitable way for everyone who has some basic information about this topic. According to our work, we are unable to find such study in the literature that has been published on this topic. Therefore, this study is conducted. Then, this study can be considered as a first step on this way and perhaps it will encourage others to provide and extend some works on this topic. The present article has been divided into two sections. In section two, we look at the pure ideals with some other types of ideals and the relations between them. In Section three, some types of rings based on the notion of pure ideals have been studied.

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2. PURE IDEALS, Z-IDEALS, STRONGLY PURE IDEALS, N-PURE IDEALS, GP-IDEALS AND Π-PURE IDEALS

The study of pure ideals is going back to around fifty years ago. The references [5-8] are the first who worked on the concept of pure ideals. This concept has been developed and studied extensively by [9-12]. Many papers have tackled this notion by different ways. In this section, our attention has been paid to study the pure ideals with some other types of ideals and discussed the relation between them. As a starting point, an ideal ζ of Ṙ is said to be pure ideal (for short PI) if, for every $v \in \zeta$ there exist $u \in \zeta$ such that $v = vu$ or equivalence to, for any ideal ξ in Ṙ, we have $\zeta \cap \xi = \zeta\xi$ is hold. A paper by [13] studied this type of ideals in the ring of (bounded) real-valued, continues functions C(Γ) on the completely regular Hawsdorff space Γ. This way may give the connection between the functional analysis and an abstract algebra. An Artin-Rees property means, for every ideal $\zeta \in$ Ṙ there exist $n \in Z^{+}$ such that $\zeta^{n} \cap \xi \subseteq \zeta\xi$, then we say that ξ has Artin-Rees property. The author proved that, the sum of every two pure ideals is also pure ideal. This result can be generalized to a finite number of pure ideals. Besides, the author considered another type of ideals namely z-ideal. This ideal is defined as follows: an ideal ζ is said to be z-ideal if, for every $v \in \zeta$ there exist $\mathcal{H}_{v}$ (the intersection of all maximal ideals in the ring which containing v) such that $\mathcal{H}_{v} \subseteq \zeta$. Clearly, the pure ideals are z-ideals. But what about the convers?. In particular, when is every z-ideal is a pure ideal in the proposed ring C(Γ)?. It's clear that from the definition of pure ideal, every pure ideal in C(Γ) has Artin-Rees property. The author answered the mentioned question as follows: z-ideal in C(Γ) is pure ideal iff it has Artin-Rees property. Further, he characterized a space Γ in which every z-ideal of C(Γ) has Artin-Rees property. He proved the following statements are equivalent: (i) Every z-ideal of C(Γ) is pure, (ii) Γ is a P-space, and (iii) Every ideal of C(Γ) is pure.

Corollary 2.1 [13] Every z-ideal of C(Γ) has Artin-Rees property iff Γ is a P-space.

Corollary 2.2 [13] Every ideal of C(Γ) has Artin-Rees property iff Γ is a P-space.

Question 1: Is there a general presentation that can give the specific number of pure ideals and z-ideals in C(Γ)?.

On the other hand, in [14] it has been defined the right (left) strongly pure ideal as follows: An ideal ζ of Ṙ is strongly right (left) pure ideal (for short R(L)SPI), if $\forall v \in \zeta$ there exist a prime element $s \in \zeta$ in which $v = vs(v = sv)$. Clearly that every strongly pure ideal is a pure ideal but the convers need not to be true. Because, the prime element s in the ideal ζ is not always existed, for example in the ring of integers module 6, we have $(3) = \{0,3\}$ is a SPI of Z_{6}. However, $(2) = \{0,2,4\}$ is a PI of Z_{6} but it's not SPI of Z_{6}. Because no prime element $s \in (2)$ in which $2 = 2 \cdot s$. Moreover, the author presented several properties of such types of ideals such as, if ζ is SPI of Ṙ, then $\xi\zeta = \xi \cap \zeta$ for every ideal ξ of Ṙ. Based on this result, it should be noted that SPI satisfying the Artin-Rees property. In other words, one can check through this direction, whether the connection between SPI and z-ideal is valid or not. Furthermore, the author proved any ideal generated by prime idempotent element is a SPI. Then, she generalized this property to a finite number of prime idempotent elements. The following proposition provided some necessary and sufficient conditions to get the SPI from PI.

Proposition 2.1 [14] Suppose Ṙ is a factorial ring with ζ be an ideal of Ṙ in which every non-zero and non-unit element of Ṙ is irreducible. Then, ζ is SPI iff ζ is PI.

Remarkably, the RSPI is not the left and the LSPI is not the right. This property has been checked in one direction by the same author. She presented a condition to get the RSPI from the LSPI. The condition which was given is to assume that Ṙ is a reduced ring. The mentioned result is given in the following proposition.

Proposition 2.2 [14] Suppose Ṙ is a reduced ring with ζ be any ideal of Ṙ. Then, ζ is a RSPI iff ζ is a LSPI.

Question 2: Is there a necessary and sufficient conditions to get the left strongly pure ideal from the right strongly pure ideal?

Question 3: Is there a general presentation that can give the specific number of strongly right (left) pure ideals in the ring Z_{n}.

Nevertheless, the author determined some other properties of SPIs which are given as follows.

Remark 2.1 Let ζ and ξ be two ideals of Ṙ. Then,

1. If ζ is SPI of Ṙ, then $\zeta \cap \xi$ is a SPI of Ṙ.
2. Let $\zeta \subseteq \xi$, then if $\zeta \cap \xi$ is SPI of Ṙ, then ζ is SPI of Ṙ.

3. The intersection of two SPIs is a SPI. This result can be generalized to a finite intersection of SPIs.
4. If ζ and ξ be two ideals of Ṙ, then ζ is SPI iff $\zeta \cap \xi$ is SPI.
5. For an arbitrary two ideals of Ṙ, if their direct sum is SPI, then one of them is SPI of Ṙ.
6. If Ṙ is a factorial ring with the direct sum of two given ideals ζ and ξ of Ṙ is SPI, then ζ and ξ are SPIs.

Proposition 2.3 [14] Let ζ and ξ be two ideals of Ṙ, if $\zeta \oplus \xi$ is SPI of Ṙ, then either ζ or ξ is SPI of Ṙ.

Corollary 2.3 [14] Let ζ and ξ be two ideals of a factorial ring Ṙ such that $\zeta \oplus \xi$ is SPI of Ṙ. Then, ζ and ξ are SPIs of Ṙ.

The concept of N-pure ideals has been introduced in [15] as a generalization for the concept of pure ideals. The motive that encouraged the author to present such types of ideals is to provide a new ring namely mid ring with some of its properties.

Definition 2.1 [15] The ideal ζ of Ṙ is said to be N-pure ideal, if for all $v \in \zeta$ there exist $u \in \zeta$ for which $v(1-u) \in \mathfrak{R}$ where $\mathfrak{R}$ is a nil-radical of Ṙ.

Clearly, any PI is a N-pure ideal and the convers is not necessarily true. Since the radical of a non-reduced ring Ṙ is a N-pure ideal but not PI. For the case that Ṙ is a reduced ring, then N-pure ideals and PIs are the same. In addition, some properties of this notion have been studied. For example, the sum of a family of N-pure ideals is a N-pure ideal. Also, the intersection (resp. product) of two N-pure ideals is a N-pure ideal. The author showed that the class of N-pure ideals is greater than the class of pure ideals. He then showed the results bellow.

Proposition 2.4 [15] Assume Ṙ be a ring. Then, Ṙ is a reduced ring iff any N-pure ideal is a PI.

Proposition 2.5 [15] Assume Ṙ be a ring and ζ is an ideal of Ṙ. Then, ζ is a N-pure ideal iff $\forall v \in \zeta\ \exists n \geq 1$ with $u \in \zeta$ such that $v^n(1-u) = 0$.

Lemma 2.1 [15] If ζ is an ideal of Ṙ. Then, ζ is a N-pure ideal iff $(\zeta + \mathfrak{R})/\mathfrak{R}$ is a PI.

Theorem 2.1 [15] Let ζ be an ideal of Ṙ. Then, the conditions bellow is equivalent.
1. ζ is a N-pure ideal.
2. For each $v_1, \dots, v_n \in \zeta$ there exist $u \in \zeta$ with $s \geq 1$ such that $v_k^s = v_k^s u$ for each $k = 1, \dots, n$.
3. For each $v \in \zeta$ there exist $s \geq 1$ such that $An(v^s) + \zeta = \dot{R}$.
4. $\sqrt{\zeta} = \{v \in \dot{R} | \exists n \geq 1, An(v^n) + \zeta = \dot{R}\}$.
5. $\sqrt{\zeta}$ is a N-pure ideal.
6. There exist a unique pure ideal ξ such that $\sqrt{\zeta} = \sqrt{\xi}$.

Corollary 2.4 [15] Assume Ṙ be a ring with ζ is an ideal of Ṙ. Then, ζ is a N-pure ideal iff ζ^n is a N-pure ideal for each $n \geq 1$.

SPI $\Rightarrow$ PI $\Rightarrow$ N-Pure Ideal

Question 4: Is there a general presentation that can give the specific number of N-pure ideals in the mid ring?

Question 5: Does the properties of strongly pure ideal are valid for a N-pure ideal?

Next, the study of [16] is one of the attempts which developed the concept of PIs. In the same year, the generalization of PIs has been presented in [17]. They defined the right (left) generalized pure ideals (for short R(resp. L)GPI) as follows: The ideal ζ of Ṙ is said to be R(L)GPI, if for each $v \in \zeta$ there is $u \in \zeta$ with $n \in Z^+$ for which $v^n = v^n u (v^n = u v^n)$. They used this generalization to present some results in [18]. They proved that for every ideal ζ of a reversible ring Ṙ, then ζ is a RGPI iff ζ is a LGPI. However, they showed that when Ṙ is a reversible ring, then $An_{\dot{R}}^r(v^n) = An_{\dot{R}}^l(v^n)$ where $v \in \dot{R}$ and $n \in Z^+$. On the other hand, the idea of right Π-pure ideals was obtained by [19] as a generalization for the RPIs. This generalization has been defined as follows: The right ideal ζ of Ṙ is called right Π-pure ideal, if for any $v \in \zeta$ there exist $u \in \zeta$ with $n \in Z^+$ and $v^n \neq 0$ such that $v^n u = v^n$. It's clear that every RPI is a right Π-pure ideal but not conversely. Furthermore, a Π-pure ideal implies GPI but not the converse. Consider the ring Z_{12}, then the ideals $(3) = \{0,3,6,9\}$ and $(4) = \{0,4,8\}$ of Z_{12} are Π-pure ideals. However, the ideal generated by (3) in Z_9 is a GPI of Z_9 but not Π-pure ideal. In addition, the author determined some condition for Π-pure ideal to be PI. The condition stated that, if Ṙ is a right SXM-ring, then every Π-pure ideal is a pure ideal, where SXM-ring is the ring in which $An_{\dot{R}}^r(v) = An_{\dot{R}}^r(v^n)$, $\forall v \neq 0 \in \dot{R}$ and $n \in Z^+$ with $v^n \neq 0$.

Question 6: Is it possible to find a relation between SPI and N-pure ideal?.

Question 7: Is there a relation between SPI and right Π-pure ideal as well as z-ideal?.

Question 8: Is there a relation between z-ideal and N-pure ideal?.

3. SOME RINGS BASED on PURE IDEALS

In the literature, there are many articles that have been written about some rings which included PIs and each one has been going in different direction. In this section, we will study in brief some of these studies and discuss them in details. In [20] the right PIP-ring has been defined. The right PIP-ring is a ring for which any principal ideal is RPI. As an illustrative example, Z_6 is a right PIP-ring. However, Z_{12} is not. Further, the author has proved that a right PIP-ring is a reduced ring and semi-prime ring. Moreover, S. H. Ahmed showed that a right PIP-ring with no zero divisors is a division ring. Then, she determined a condition under which, the right PIP-ring becomes a division ring. This condition states that, if Ṙ is a right uniform PIP-ring, then Ṙ is a division ring. Moreover, a paper by Husm [21] generalized the results of [20] to present the right PIGP-ring which is a ring whose principal ideals are GPIs. The ring Z_{12} is a PIGP-ring. It is clear that, any right PIP-ring is a right PIGP-ring but not conversely. The author obtained a condition under which the convers is true. In particular, H. Q. Mohammad proved that the right PIGP-ring is right PIP-ring whenever $An^r_{\dot{R}}(v^n) \subseteq An^r_{\dot{R}}(v), \forall v \in \dot{R}$ and $n \in Z^+$. The mentioned result is stated bellow.

Theorem 3.1 [21] Let Ṙ be a right PIGP-ring and $An^r_{\dot{R}}(v^n) \subseteq An^r_{\dot{R}}(v)$ for each $v \in \dot{R}$ with $n \in Z^+$. Then, Ṙ is a PIP-ring.

However, he proved under the same condition, the right PIGP-ring is a regular ring. Moreover, the author characterized the uniform PIGP-ring in the terms of nilpotent and non-zero divisor. This result is given as follows.

Theorem 3.2 [21] Let Ṙ be a zero-commutative uniform ring. Then, Ṙ is a PIGP-ring iff:

1. For each $v \in \dot{R}$ is either nonzero divisor or nilpotent.
2. Any nonzero divisor of Ṙ is an invertible.
3. $N(\dot{R})$ is the right ideal in Ṙ.

Theorem 3.3 [21] Let Ṙ be a PIGP-ring and $An^r_{\dot{R}}(v^n) \subseteq An^r_{\dot{R}}(v)$ for each $v \in \dot{R}$ with $n \in Z^+$. Then, Ṙ is a regular ring.

Theorem 3.4 [21] Let Ṙ be a commutative PIGP-ring. Then, $3\mathrm{Rad}(v\dot{R}) = v\dot{R} + N$, where N is the nil radical of Ṙ.

Two years later, a right PILP-ring has been introduced in [22]. A right PILP-ring is a ring in which any principal right ideal is a LPI. For instance, Z_6 is a right PILP-ring since the ideals generated by (2) and (3) are LPIs of Z_6. But Z_{12} is not a right PILP-ring, because the ideal generated by 2 is not PI of Z_{12}. In addition, the author showed that Ṙ is a reduced ring whenever Ṙ is an abelian right PILP-ring. Furthermore, R. D. Mahmood also proved, if Ṙ is an abelian right PILP-ring with $An^l_{\dot{R}}(v) = 0$, then Ṙ is a division ring. It should be noted that, the right PIP-ring and the right PILP-ring are different. Since the first is considered the right principal ideal is pure in general. However, the second is concentrated on the left pure ideal. Thereafter, the right Nil-pure ring has been introduced in [23]. This ring was defined as follows: The ring Ṙ is said to be right Nil-pure ring, if for any $v \in N(\dot{R})$, we have $An^r_{\dot{R}}(v)$ is a LPI of Ṙ. A ring of integers Z is a right Nil-pure ring but Z_{12} is not, since the ideal generated by 6 in Z_{12} is not PI. The authors proved some good results which are as follows: If Ṙ is a right Nil-pure ring with the ideal ζ is a PI of Ṙ, then $\dot{R}/\zeta$ is a Nil-pure ring. Further, for the regularity of this ring, they showed that whenever Ṙ is a right Nil-pure ring, then the center of Ṙ contains no non-zero nilpotent element. Moreover, they concluded that the center of this ring is a reduced. Thereafter, they proved that, if Ṙ is a reversible and Nil-pure ring, then Ṙ is a reduced ring. Also, they concluded that, if Ṙ is a reversible and right Nil-pure ring, then Ṙ is a left (right) non-singular ring. Finally, they presented the following characterization.

Theorem 3.5 [23] Let Ṙ be a reversible ring. Then, Ṙ is a Nil-pure ring iff Ṙ is n-regular ring.

Later on, in [24] new ring has been presented namely JGP-ring. This ring is defined as follows, Ṙ is JGP-ring if for every $v \in J(\dot{R})$, then $An^r_{\dot{R}}(v)$ is a LGPI, where $J(\dot{R})$ is a Jacobson radical of Ṙ. They proved the following results.

Theorem 3.6 [24] Let Ṙ be a right JGP-ring with ζ is a PI of Ṙ. Then, $\dot{R}/\zeta$ is JGP-ring.

Theorem 3.7 [24] Let Ṙ be a reversible ring. Then, Ṙ is a right JGP-ring iff $An^r_{\dot{R}}(v)+An^r_{\dot{R}}(u^n) = \dot{R}$ for each $v \in J(\dot{R})$ and $u \in An^r_{\dot{R}}(v)$ with $n \in Z^+$.

Some studies have discussed the maximal ideals and presented some good results. For instance, the study of [25] presented MRGP-ring. It is a ring whose maximal left ideals are RGPIs. i.e., let ζ be a maximal ideal of Ṙ with $v \in \zeta$, then ζv is a RGPI. Further, they discussed the connection between this ring and MRCP-ring which is a ring such that all of its maximal left ideals are right Co-pure ideals (see [26] for details). Moreover, it is clear that every MRCP-ring is an MRGP-ring but the convers need not to be true. Since the right Co-pure ideal implies right GP-ideal but not conversely. The authors presented under some conditions; the MRGP-ring can be MRCP-ring. Their result is given bellow.

Theorem 3.8 [25] Let Ṙ be a reduced, MRGP-ring. Then, Ṙ is an MRCP-ring.

Furthermore, they have shown that under the same hypothesis of the above theorem, the ring can be strongly regular ring. Also, it was shown in [25] that, if Ṙ is a right uniform reduced MRGP-ring, then Ṙ is a division ring. Further, if Ṙ is a zero-commutative, MRGP-ring, then Ṙ is kasch ring. The kasch ring is the ring for which every maximal right (left) ideal is a right (left) annihilator, (see [27] for details). On the other hand, it has been defined in [28] a ring such that its maximal right (left) ideal is a L(R)GPI. Such ring is said to be right (left) MGP-ring. Clearly that every MRGP-ring is MGP-ring. They proved that, if Ṙ is reduced, MGP-ring then $An^r_{\dot{R}}(u^n)$ is a direct sum of Ṙ for each $v \in$ Ṙ and $n \in Z^+$. Further, if Ṙ is reduced, MGP-ring, then Ṙ is kasch ring. Besides that, they provided the following condition: Let $u \in$ Ṙ with $n \in Z^+$ such that $An^l_{\dot{R}}(u^n) \subseteq An^r_{\dot{R}}(u)$ then, under this condition, Ṙ is right MGP-ring iff it's a strongly regular ring. Also, if Ṙ is a right MGP-ring which satisfying the mentioned condition with $v^n u = 0$ where $v, u \in$ Ṙ and $n \in Z^+$, then $An^r_{\dot{R}}(v^n) + An^r_{\dot{R}}(u) =$ Ṙ. Moreover, the authors showed that under the same condition with Ṙ is a right MGP-ring, then Ṙ is a reduced weakly regular ring. Thereafter, they proved that Ṙ is a division ring whenever Ṙ is uniform semi-commutative, MGP-ring and each of its ideal is principal. Ultimately, they defined in [29] new ring namely MEP-ring which is a ring its maximal essential right ideal is a LPI. They determined the following results.

Theorem 3.9 [29] Let Ṙ be a semi-commutative, right MEP-ring. Then, Ṙ is a reduced ring.

Theorem 3.10 [29] Let Ṙ be a semi-commutative, right MEP-ring. Then, every essential right ideal of Ṙ is an idempotent.

Moreover, they discussed another important result concerned with MEP-ring. In particular, they showed that, if Ṙ is a semi-commutative, right MEP-ring, then $J(\dot{R}) = (0)$. Besides, they obtained that under the same hypothesis, Ṙ becomes reduced weakly regular ring. Finally, they proved the following theorem.

Theorem 3.11 [29] The ring Ṙ is a strongly regular iff it's a semi-commutative, MEP, MERT-ring. The MERT-ring is a ring such that each of its maximal essential right ideal is a two-sided ideal, (see [30] for details).

Raida and Shahla defined in [31] the EGP-ring. This ring is a ring such that each of its essential right ideal is a left GP-ideal. The authors proved the next results. If Ṙ is an ERT-ring then, (i) Ṙ is a fully left idempotent ring, (ii) Ṙ is a right EGP-ring, where, ERT-ring is a ring in which each of its essential right ideal is a two-sided ideal [32]. The recent study has been presented the MLGP-ring in [33]. This ring was defined to be a ring for which any right maximal ideal is a LGPI. Further, they showed that, if Ṙ is a local, SXM and NJ-ring, then Ṙ is a strongly π-regular ring iff Ṙ is MLGP-ring, where NJ-ring is a ring in which $N(\dot{R}) \subseteq J(\dot{R})$. Besides, they also proved the following results: If Ṙ is a local, SXM and MLGP-ring, then (i) $J(\dot{R}) = 0$, (ii) if Ṙ is an NJ-ring, then $An^r_{\dot{R}}(v^n)$ is the direct sum for each $v \in$ Ṙ and $n \in Z^+$. Furthermore, the Mid-ring has been defined in [15], which is a ring in which for any $v \in$ Ṙ, we have $An_{\dot{R}}(v)$ is a N-pure ideal. Further, the author showed that, when Ṙ is a Mid-ring and the ideal ζ is a PI of Ṙ, then $\dot{R}/\zeta$ is a Mid-ring. Besides that, he proved that, if Ṙ is a Mid-ring for which $\dot{R} = \prod_m \dot{R}_m$, then each of $\dot{R}_m$ is a Mid-ring. Let Ṙ be a ring, then Ṙ is called primary ring whenever its zero ideal is a primary ideal. By using the concept of primary ring, the author presented the characterization for the Mid-ring which is as follows.

Theorem 3.12 [15] Let Ṙ be a ring. Then, the following axioms are equivalent.

1. Ṙ is a Mid-ring.
2. If $vu = 0$, $\exists n \geq 1$ such that $An_{\dot{R}}(v) + An_{\dot{R}}(u^n) =$ Ṙ.
3. $\dot{R}_q$ is primary ring for each $q \in$ Spec(Ṙ).
4. $\dot{R}_t$ is primary ring for each $t \in$ Max(Ṙ).
5. $Ker\pi_q$ is PI for any $q \in$ Spec(Ṙ).
6. $Ker\pi_q$ is PI for any $q \in$ Min(Ṙ).

7. $Ker\pi_q = Ker\pi_p$ in which q, p are prime ideals with $q \subseteq p$.
8. $Ker\pi_q$ is primary ideal for each $q \in \mathrm{Spec}(\dot{\mathrm{R}})$.
9. $Ker\pi_t$ is primary ideal for each $t \in \mathrm{Max}(\dot{\mathrm{R}})$.

Theorem 3.13 [15] Suppose p be a prime ideal of a Mid-ring $\dot{\mathrm{R}}$. Then, p is a N-pure ideal iff $p \in \mathrm{Min}(\dot{\mathrm{R}})$.

Theorem 3.14 [15] Every mid-ring is mp-ring.

Furthermore, the author obtained that any Gpf-ring is a Mid-ring, where the Gpf-ring is the ring in which each $v \in \dot{\mathrm{R}}, \exists\, n \geq 1$ in which $An_{\dot{\mathrm{R}}}(u^n)$ is a PI. Finally, we end this section by the following results. In [16], the author presented the concept of right (left) MP-ring which is a ring contains every maximal right (left) ideal is a L(R)PI. Then, some new results concerned such types of rings have been introduced in [34]. In brief, these results are given as follows: (i) Let $\dot{\mathrm{R}}$ be a right MP-ring with $An^r_{\dot{\mathrm{R}}}(v)$ is a weak ideal of $\dot{\mathrm{R}}$ for each $v \in \dot{\mathrm{R}}$. Then, $\dot{\mathrm{R}}$ is a strongly regular ring, (ii) let $\dot{\mathrm{R}}$ be a right MP-ring with $An^r_{\dot{\mathrm{R}}}(v)$ is a weak ideal of $\dot{\mathrm{R}}$ for each $v \in \dot{\mathrm{R}}$. Then, $\dot{\mathrm{R}}$ is a right division ring if it's an uniform ring.

4. CONCLUSINS

As a conclusion, we have seen from the works of many researchers that the notion of PIs was very important in the concept of ring theory. Based on this notion, many articles have been presented and considered this notion in different way. The present study showed that PIs with their generalizations have an important role in determining the structure of a new ring and this can be noted in the study of some properties about it and its relations with some other types of rings. Therefore, the present study concluded the following points:

1. PIs gives N-pure ideals but not convers.
2. PIs gives z-ideals and the convers is true with some necessary and sufficient conditions.
3. SPIs implies PIs but not conversely. It should be noted that, PIs lies between SPIs and N-pure ideals.
4. RPI implies right Π-pure ideal but not convers.
5. Right Π-pure ideal gives GPI but not converse.
6. PIP-ring, PIGP-ring and PILP-ring have been introduced based on the principal ideals with PIs and GPIs.
7. Nil-pure-ring and JGP-ring have been provided based on the concepts of nil radical, annihalitor and Jacobson radical.
8. Based on the maximal ideals with PIs, MRGP-ring, MRCP-ring, right (left) MGP-ring and right (left) MP-ring have been presented.
9. Based on the concept of N-pure ideals, an algebraic structure called Mid-ring has been introduced.

5. REFERENCES

1. Nicholson, W. K. and Zhou, Y. (2005). " Clean Rings: A Survey", *In Advances in ring theory*, vol.12, no. 1, pp. 1-18. DOI: 10.1142/9789812701671_0017
2. Yang, A X. (2009). "A Survey of Strongly Clean Rings", *Acta. Appl. Math.* vol. 108, pp.157-173. DOI https://doi.org/10.1007/s10440-008-9364-6
3. Srivastava, A. K. (2010). "A Survey of Rings Generated by Units", *Annales de la Facult´e des Sciences de Toulouse*, vol. 19, no.1, pp. 203-213.
4. Dixon, M. R, Kurdachnko, L. A. and Pypka, A. A. (2015). "The theorems of Schur and Baer: A Survey", *International Journal of Group Theory*, vol. 4, no. 1, pp.21-32. DOI: 10.22108/IJGT.2015.7376
5. Lazard, D. (1967). "Disconnexités des spectres d'anneaux et des préschémas", *Bull. Soc. Math. France*, vol. 95, pp.195-108. DOI: 10.24033/bsmf.1649
6. Fieldhouse, D. J.(1969). "Pure Theories", *Math. Ann*, vol. 184, pp. 1-18. DOI: https://doi.org/10.1007/BF01350610
7. Bkouche, R. (1970). "Pureté, mollesse et paracompacité", C. R. Acad. Sci. Paris Sér. A-B, 270.
8. Fieldhouse, D. J. (1971). "Regular Modules Over Semi Local Rings", Coll. Math. Sco. Janos Bol-yai, pp. 193-196.
9. Marco, G. De. (1983). "Projectivity of Pure Ideals", Rend. Sem. Mat. Univ. Padova, vol. 69, pp. 289-304.
10. Al-Ezeh, H. (1988). "The Pure Spectrum of PF-rings", *Commutatively Math*. University S. P. vol. 37, no. 2, pp. 179-183.
11. Al-Ezeh, H. (1989). "Pure Ideals in Commutative Reduced Gelfand Rings with Unity", *Arch. Math*. vol.53, pp. 266-269. https://doi.org/10.1007/BF01277063
12. Al-Ezeh, H. (1989). "On Generalized PF-rings", *Math. J. Okayama Uni*. Vol. 31,pp. 25-29.
13. Afrooz. S. (2016). "Pure Ideals, z-ideals and Ideals with Artin-Rees Property in C(Γ)", 4th Seminar on Algebra and its Applications. August 9-11, University of Mohaghegh Ardabili.
14. Abdullah, N. K. (2015). "Strongly Pure Ideals and Strongly Pure Sub-modules", *Kirkuk University Journal of Scientific Studies*, vol. 10, no 1, pp. 12-28.
15. Aghajani, M.(2020) "N-pure Ideals and Mid Rings", arXiv preprint arXiv:2011.02134.
16. Mahmood, R. D.(2000). "On Pure Ideals and Pure Submodules", Ph. D. Thesis Mosul university.
17. Shuker, N. H. and Mahmood, R. D. (2000) "On Generalization of Pure Ideals", *Journal of Education and Science*, vol. 43, pp. 86-90.
18. Mahmood, R. D. and Khalil, S. M. (2009)."On GP-Ideals", *Rafidain Journal of Computer Science and Mathematics*. vol. 6, no. 3, pp. 57-63.
19. Ahmad, S. H.(2014). "On Π-pure Ideals", *Rafidain Journal of Computer Science and Mathematics*, vol.

11, no. 2, pp. 83-86. DOI: 10.33899/csmj.2014.163751

20. Ahmad, S. H. (2006) "On Rings Whose Principal Ideals are Pure", *Rafidain journal of computer science and mathematics*, vol. 3, no. 2, pp. 53-57.
21. Mohammad, H. Q.(2007). "On Rings Whose Principal Ideals are Generalized Pure Ideals", *Rafidain Journal of Computer Science and Mathematics*, vol. 4, no. 1, pp. 63-68.
22. Mahmood, R. D. (2009). "PILP-rings and Fuzzy Ideals", *Rafidain Journal of Computer Science and Mathematics*, vol. 6, no. 2, pp. 31-37. DOI: 10.33899/csmj.2009.163794
23. Hajo, A. M. and Mahmood, R. D.(2017). "On Nill-pure Rings", *International Journal of Enhanced Research in Science, Technology and Engineering*, vol. 6, no.11, pp. 25-28.
24. Majeid, E. S. and Mahmood, R. D. (2019). "JGP-Ring", *Open access library journal*, vol. 6, e5626, pp. 1-4. DOI: 10.4236/oalib.1105626
25. Mahmood, R. D. and Abd, M. A. (2005). "On Rings Whose Maximal Ideals are GP-ideals", *Rafidain Journal of Computer Science and Mathematics*, vol. 2, no. 2, pp.27-31.
26. Shuker, N. H. and Mahmood, R. D. (2000). "On Rings Whose Maximal Ideals are Co-pure", *Rafidain Journal of Computer Science and Mathematics*, vol. 11, no. 4.
27. Ming, R. Y. C. (1986). "On Semi Prime and Reduced Ring", *Riv. Math. Univ. Parama*, vol. 4,no.12, pp. 167-175.
28. Mahmood, R. D. and Mahmood, A. B. (2008). "Maximal Generalization of Pure Ideals", *Rafidain Journal of Computer Science and Mathematics*, vol. 4, no. 1,pp. 57-62.
29. Mahmood, R. D. and Mahmood, A. B. (2007). "On Rings Whose Maximal Essential Ideals are Pure", *Rafidain Journal of Computer Science and Mathematics*, vol. 5, no. 1,pp. 21-27. DOI: 10.33899/csmj.2007.163995
30. Ming, R. Y. C. (1983). "Maximal Ideals in Regular Rings", *Hokkaido Mathematical Journal*, vol. 12, pp. 119-128.
31. Mahmood, R. D. and Khalil, S. M. (2010). "MGP and EGP- rings", *Rafidain Journal of Computer Science and Mathematics*, vol. 7, no. 2, pp.59-65. DOI: 10.33899/csmj.2010.163885
32. Brown, S. H. (1973). "Rings Over Which Every Simple Module is Rationally Complete", *Canad. J. Math*, vol. XXV, no. 4, pp. 693-701. DOI: https://doi.org/10.4153/CJM-1973-070-0
33. Mahmood, R. D. and Mageed, E. S. (2019). "On MLGP-rings", Rafidain Journal of Computer Sci-ence and Mathematics, vol. 13, no. 2, pp.61-66. DOI: 10.33899/csmj.2020.163521
34. Mahmood, R. D. and Hajo, A. M. (2020). "On MP-rings", *Rafidain Journal of Computer Science and Mathematics*, vol. 14, no. 1, pp. 35-40. DOI:10.33899/csmj.2020.164797

Arabic Abstract

تعتبر نظرية الحلقات احد فروع الجبر المجرد الذي يركز على دراسة نظام رياضي مكون من عمليتين ثنائيتين في اتجاهات مختلفة. هناك الكثير من الابحاث في هذا المجال قدمت نتائج جديدة حول بنية جبرية معينه. مفهوم المثاليات النقية من المفاهيم المهمة في هذا المجال والذي تمت دراسته قبل اكثر من 40 سنة حول بعض الحلقات وكذلك بعض المقاسات. في هذا البحث قمنا بمراجعة مختصرة حول المثاليات النقية والمثاليات النقية المعممة و علاقاتها مع بعض المثاليات الاخرى مثل المثاليات النقية من النوع-N ، مثالي من نوع-Z ، مثالي نقي من نوع-Π و المثاليات النقية القوية. كذلك تم تلخيص اهم النتائج التي قدمت حول هذا الموضوع مع طرح بعض الاسئلة التي تشير الى انه هل بالإمكان ايجاد خواص جديدة من خلال دراسة العلاقة بين تلك المثاليات والمثاليات الاخرى. اخيرا، نتيجة هذه الدراسة اوضحت اهمية المثاليات النقية وتعميماتها والدور الذي تلعبه في تكوين البنية الجبرية وخواصها وعلاقتها مع بنى جبرية اخرى.