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A Study on Optimal Approximation of Functional Subsets within Real Normed Spaces

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ABSTRACT

The optimal approximation of functional subsets within standard spaces facilitates data modeling and management of linear and nonlinear systems. In this paper, the best approximation in a real standard linear space X is described by the Kolmogorov theorem. In addition, the concepts of proximal set, smooth space, sun, sun point, and their relationship with the Kolmogorov condition are discussed. Finally, the effectiveness of using the best approximation in practical situations to achieve high accuracy in the computation of standard linear spaces is highlighted.

NOMENCLATURE	
W	a topological space that is compact and Hausdorff
W(T)	the space of real continuous functions on T
x	Real Normed Spaces
T	T is a compact Hausdorff topological space
crit	Critical Point

1. INTRODUCTION

In recent decades, there has been an increasing need to study the importance of best approximation within the field of functional analysis and in particular within the field of standard linear spaces, [1-4]. This study seeks to provide a detailed description of the best approximation in real normed linear space X through a specialized theorem that highlights the main aspect of the theoretical trends. Furthermore, the fundamental concepts within this trend related to close sets, suns and sun points, how these concepts are interconnected, and their importance for Kolmogorov's condition are investigated [5]. This provides comprehensive coverage of approximation theory, practical effect, and its extension into real standard spaces. The following will be the setting for this paper: We refer to W to be as a topological space that is compact and Hausdorff [6]. W(T) will denote the space of real continuous functions on T unless explicitly stated, and the complex space W(T) is also being considered. The uniform standard is installed in the spaces W(T). Let B be a non-empty subset of W(T). The uniform norm is defined by nown as the distance between u and B. Therefore,

$$\| u \| = \max_{t \in \mathcal{T}} |u(t)| \quad \text{for all } u \in W(T)\,, \qquad (1)$$

And

$$d(u\,,\,B) = \inf_{h \in B} \| u - h \| \quad \text{for} \quad u \in W(\mathcal{T})\,, \qquad (2)$$

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$$P_B(u) = \{h \in B : \| u - h \| = d(u, B)\}, \quad (3)$$

As a collection of the best approximations of u from B. Now given $u \in W(\mathcal{T})$, the main problem is finding an element $h_0 \in B$ such that $\| u - h_0 \| \leq \| u - h \|$, for all $h \in B$. Such an element, if present, is called the best approximation element (closest element) to u from B. The number $\| u - h_0 \|$ is then the distance from u to B and $d(u,B) = \| u - h_0 \|$. The best approximation when $P_B(u)$ is non-empty? (the members of $P_B(u)$ will be called best approximation to u from B), that is, which properties of B and the space ensures that $P_B(u) \neq \emptyset$ for each $u \in W(\mathcal{T})$, in this case, B is said to be proximinal in $W(\mathcal{T})$. This is obtained by the following theorem:

Theorem 1: [6]. Let B be a finite dimensional subspace of a normed linear space $\mathcal{X}$. Then u or each $u \in X$, such that $\exists$ an element in B of best approximation to u.

Another characterization of best approximation in $W(\mathcal{T})$ is given by the following theorem .

Theorem 2 .[7]. A function h_0 in B is the best approximation to $u \in W(T) \Leftrightarrow \forall h \in B$

$$\max_{t \in \mathcal{T}_0} \mathrm{Re}\left\{[u(t) - h_0(t)]\overline{h(t)}\right\} \geq 0$$

where $\mathcal{T}_0 = \mathrm{crit}(u - h_0)$.

Lemma 1. Let $R = \{e + \lambda(x - e) : \lambda \geq 0\}$. Then

$$\bigcup_{y \in R} B(y, \| y - e \|) = \mathrm{int}\bigcap\{\varphi^{-1}((-\infty, \varphi(e)]) : \varphi \in \mathcal{X}^*, \varphi(e - x) = \| \varphi \| \| e - x \|\} = \bigcap\left\{ \varphi^{-1}\left(\left(-\infty, \varphi(e)\right)\right) : \varphi \in \mathcal{X}^*, \varphi(e - x) = \| \varphi \| \| e - x \| \right\}.$$

Proof: Suppose B be a magnification with center s, that is,

$$B: \mathcal{X} \to \mathcal{X}$$
$$B(t) = e + \lambda(t - e), \text{ for } t \in \mathcal{X},$$

where $\lambda > 0$. B is a bijection. On the other hand, the open ball $\mathfrak{B}(x, \| x - e \|)$ and the single point set $\{e\}$ are two disjoint convex sets. So there exists (by the separatior theorem, [8]) $\varphi \in \mathcal{X}^*$ such that

$$\varphi(e - x) = \| \varphi \| \| e - x \|$$

$\varphi(t) \leq \varphi(e)$, for all $t \in \mathfrak{B}(x, \| x - e \|)$.

Since $B\left(\mathfrak{B}(x, \| x - e \|)\right) = \mathfrak{B}(B(x) , \| x - e \|)$ then for each $z \in \mathfrak{B}(B(x), \lambda \| x - e \|), z = M(t) = s + \lambda(t - e)$, for some $t \in \mathfrak{B}(x, \| x - e \|)$. It follows that $\varphi(z) = \varphi(e) + \lambda\varphi(t - e)(t - e) \leq \varphi(e)$. So for each y in R and $z \in \mathfrak{B}(y, \| y - e \|), z \in \varphi^{-1}\left(\left(-\infty, \varphi(e)\right)\right)$, for each $\varphi \in \mathcal{X}^*$ for which $\varphi(e - x) = \| \varphi \| \| e - x \|$. That is

$$\bigcup_{y \in R} \mathfrak{B}(y, \| y - e \|) \subset \mathrm{int}\bigcap\{ \varphi^{-1}((-\infty, \varphi(e)]) : \varphi \in \mathcal{X}^*, \varphi(e - x) = \| \varphi \| \| e - x \| \} \subset \bigcap\left\{ \varphi^{-1}\left(\left(-\infty, \varphi(e)\right)\right) : \varphi \in \mathcal{X}^*, \varphi(e - x) = \| \varphi \| \| e - x \| \right\}.$$

Now suppose that $z \notin \bigcup_y \in R\mathfrak{B}(y, \| y - e \|)$. Thus the open set $\bigcup_y \in R\mathfrak{B}(y, \| y - e \|)$ and line scgment $[z, e]$ are two disjoint convex sets and so by separation thcorem thero exists $\varphi \in \mathcal{X}^*$ that is $\varphi(e - x) = \| \varphi \| \| e - x \|$ which separate two sets $[z, e]$ and $\bigcup_{y \in R}\mathfrak{B}(y, \| y - e \|)$, i.e., $z \notin \varphi^-1\left(\left(-\infty, \varphi(e)\right)\right)$ and so

$$z \notin \bigcap\left\{ \varphi^{-1}\left(\left(-\infty, \varphi(e)\right)\right) : \varphi \in \mathcal{X}^*, \varphi(e - x) = \|\varphi\|\|e - x\| \right\} \quad \blacksquare$$

2. KOLMOGROV'S AND SUNS DESCRIPTION

Let B is a subset of $\mathcal{X}$ that is not empty and (B may not be a linear subspace of $\mathcal{X}$). If $x \in \mathcal{X} \setminus$ Band $e \in P_B(x)$, it is always true that $e \in P_B(y)$,for $y = e + \lambda(x - e)$, for all $\lambda \in [0,1]$ (since $y = \lambda x + (1 - \lambda)$ e then

$$\| x - y \| + \| y - e \| = (1 - \lambda) \| x - e \| + \lambda\|x - e\| = \|x - e\|$$

and for each $h \in$ B it follows that

$$\| y - h \| \geq \| x - h \| - \| x - y \| \geq \| x - e \| - \| x - y \| = \| y - e \| ,$$

that is, $e \in P_B(y)$). The point e is said to be a Solar point in B for x,if $e \in P_B(y)$ for every $y = e + \lambda(x - e)$, for $\lambda \in (1, \infty)$. That is, e is a Solar point in B for x, if $e \in P_B(y)$, for every y in the half-line $R = \{e + \lambda(x - e) : \lambda \geq 0\}$. A set B is said to be a sun in $\mathcal{X}$, if for each $x \in \mathcal{X} \setminus$ B,the set $P_B(x)$ contains a Solar point for x and the set R denotes a ray of the sun which passes through x. There are numerous variant concepts:

Alpha-sun, beta-sun, gamma-sun and delta -sun, meta suns and strict suns. The concept of `sun' seems to be the most important. It was formulated in the $50'$s by Efimov and Stechkin [8]. It is important by the reason of results such as the following theorem.

Theorem 3: [9,10]. **(Vlasov):** If $\dim \mathcal{X} < \infty$ and M is a Chebyshev subset of X then B is a sun.

A space $\mathcal{X}$ is said to be Smooth if for each $x \in E(\mathcal{X})$ (the sphere $E_{\mathcal{X}}(0,1)$ in $\mathcal{X}$) there is a unique hyperplane of support to $E(\mathcal{X})$ at x. The condition is equivalent to the condition that the norm $\|.\|$ be Gateaux differentiable at each point of $\mathcal{X}\setminus\{0\}$.

Theorem 4. If X is a Smooth space and B is a sun in X then B is convex.

Proof: Suppose that $x \notin \mathrm{B}$, so there is a Solar point in B for x,say $e \in \mathrm{B}$. Ther $\mathfrak{B}(x, \| x-e \|) \cap \mathrm{B} = \emptyset$ and for each y in the half-line $R = \{e + \lambda(x-e): \lambda \geq 0\}$, it follows that $B(y, \| y-e \|) \cap \mathrm{B} = \emptyset$. So $\left(\bigcup_{y \in R} B(y, \| y-e \|)\right) \cap \mathrm{B} = \emptyset$. Since $\mathcal{X}$ is a Smooth space then $\bigcup_{y \in R} \mathfrak{B}(\mathrm{y}, \| \mathrm{y} - \mathrm{e} \|) = H_x$ is an open half- space (by lemma 1) . So its complement is a closed half-space containing B and not x. Let $U_x = \mathcal{X} \backslash H_x$. The intersection over all $x \notin \mathrm{B}$ of these closed half-spaces is convex and equal to B,that is , $\bigcap_x \notin \mathcal{MF}_x = \mathrm{B}$ is convex. ■

The following theorem is derived from Theorems 1 and 2 as well as Vlasov's Theorem [10].

Theorem 5: If $\dim \mathrm{X} < \infty$ and X is a Smooth space then a Chebyshev subset of X is a closed convex set.

The concept of Solar point is what one needs to make sense of Kolmogorov's Characterization of best approximation.

Theorem 6. Suppose that $x \in \mathrm{X} \setminus \mathrm{B}$ and $\mathrm{h}_0 \in \mathcal{M}$. Then the following facts are equivalent:

1. $h_0 \in P_{\mathrm{B}}(x)$ and h_0 *is* a Solar point for x *in* B
2. $[h_0, h] \cap B(x, \| x - h_0 \|) = \emptyset$, for all $h \in \mathrm{B}$ ($that\ is$, $h_0 \in P_{[h_0,h]}(x)$, for each $h \in \mathrm{B}$).
3. 3-For any $\mathrm{h} \in \mathrm{B}$,there exists $\varphi \in \mathrm{ext}\ E(\mathrm{X}^*)$ such that $\varphi(h_0 - x) = \| h_0 - x \|$,

$$\varphi(h) \geq \varphi(h_0) ,$$

Where $E(\mathrm{X}^*)$ is the unit sphere in X^*.

4. each $\mathrm{h} \in \mathrm{B}$, there exists $\varphi \in \mathrm{E}(\mathrm{X}^*)$ such that $\varphi(\mathrm{h}_0 - \mathrm{x}) = \| \mathrm{h}_0 - \mathrm{x} \|$, $\varphi(\mathrm{h}) \geq \varphi(\mathrm{h}_0)$.

It is worth noting that the third condition of the above theory represents the abstract form of Kol- mogorov's condition. Let $\mathcal{X} = \mathrm{W}(T)$, where T is a compact Hausdorff topological space, B is a subspace of $\mathrm{W}(\mathcal{T})$ and $f \in \mathrm{W}(\mathcal{T}) \setminus \mathrm{B}$,since $\mathrm{W}(\mathcal{T})^* \cong \mathcal{M}(\mathcal{T})$, the space of regular Borel measure on $\mathcal{T}$, part (3) implies that, for each $g \in \mathrm{B}$, there exists measure $\mu \in \mathrm{ext}\ E(\mathrm{B}(\mathcal{T})) \cong \mathrm{ext}\ E(\mathcal{X}^*)$ such that :

$$\begin{cases} \mu(h_0 - u) = \|h_0 - u\|, \\ \mu(h) \geq \mu(h_0). \end{cases}$$

On the other hand,

$$\mathrm{ext}\ E(\mathrm{B}(\mathcal{T})) = \{\mu \in \mathrm{B}(\mathcal{T}) :\ \pm\mu(\{t\}) = \| \mu \| = 1 \text{ for some } t \in \mathcal{T}\} = \{\mu \in \mathrm{B}(\mathcal{T}) : \| \mu \| = 1 , |\operatorname{supp} \mu| = 1\}$$

$$= \{\mu = \pm e(t) :\ t \in \mathcal{T}\ \} ,$$

where $e(t) = \delta_t$ is evaluation functional. This implicitly means that:

$$\varepsilon(h_0 - u)(t) = \| h_0 - u \| ,$$

$$\varepsilon(h - h_0)(t) \geq 0 ,$$

where $\varepsilon \in \{-1,1\}$. It follows that

$$|h_0(t) - u(t)| = \| h_0 - u \| ,$$

$$\big(h_0(t) - u(t)\big)\big(h(t) - h_0(t)\big) \geq 0 .$$

Thus, by the notation of Kolmogorov's theorem, $t \in \mathcal{T}_0$ and for $h_1 = h_0 - h \in \mathrm{B}$

$$[u(t) - h_0(t)]h_1(t) \geq 0 ,$$

That is, in the real case, $\max_{t \in T_0} [u(t) - h_0(t)]h_1(t) \geq 0$, is satisfied.

Theorem 7. Let B be a finite dimensional subspace of $\mathrm{W}(\mathcal{T})$. If $u \in \mathrm{W}(\mathcal{T}) \backslash \mathrm{B}$ and $h_0 \in \mathrm{B}$, then, the conditions listed in the following points are equivalent:

1. $\mathrm{h}_0 \in \mathrm{P}_\mathrm{B}(\mathrm{u})$,
2. Kolmogorov' s condition:

$$\max_{\mathrm{t} \in \mathrm{T}_0} \mathrm{Re}\left\{[\mathrm{u(t)} - \mathrm{h}_0(\mathrm{t})]\overline{\mathrm{h(t)}}\right\} \geq 0 ,$$

for each $\mathrm{h} \in \mathrm{B}$, where $\mathcal{T}_0 = \mathrm{crit}(\mathrm{f} - \mathrm{h}_0)$,

3.The condition of the complex Characterization Theorem:

$$0 \in \mathrm{co}\left\{\left[\overline{\mathrm{u(t)} - \mathrm{h}_0(\mathrm{t})}\right] \mathrm{v(t)} :\ \mathrm{t} \in \mathrm{crit}(\mathrm{u} - \mathrm{h}_0)\right\} ,$$

4.There exists a non- empty finite subset $\mathrm{A} = \{\mathrm{t}_1, \dots, \mathrm{t}_\tau\}$ of $\mathcal{T}$ and there is a non-zero value $\alpha(\mathrm{t})$ for $\mathrm{t} \in \mathrm{A}$ with $\sum_{\mathrm{t} \in \mathrm{A}} |\alpha(\mathrm{t})| = 1$, such that

$$\sum_{\mathrm{t} \in \mathrm{A}} \alpha(\mathrm{t}) \mathrm{v(t)} = 0 ,$$

and

$$\sum_{\mathrm{t} \in \mathrm{A}} \alpha(\mathrm{t}) [\mathrm{u(t)} - \mathrm{h}_0(\mathrm{t})] = \|\mathrm{u} - \mathrm{h}_0\| ,$$

5.There exists a non-empty finite subset $A = \{t_1, \dots, t_r\}$ of $\mathcal{T}$ and there exists a non-zero $\alpha(t)$ for $t \in A$ with $\sum_{t\in\mathcal{A}} |\alpha(t)| = 1$ such that we obtain

$$f(t) - h_0(t) = \sigma(t) \parallel u - h_0 \parallel , \; for\; t \in A ,$$

where $\sigma(t) = \text{sgn}\, \alpha(t), for\; t \in A$.

Proof: The equivalence $(1) \Leftrightarrow (2)$ and $(a) \Leftrightarrow (c)$ are in Kolmogorov's'Characterization implies that $(1) \Leftrightarrow (5)$. We show that $(4) \Leftrightarrow (5)$. Now assume that (4) holds. The equality in above theorem implies that

$$\parallel u - h_0 \parallel = \left| \sum_{t\in A} \alpha(t)[u(t) - h_0(t)] \right| \leq \sum_{t\in A} |\alpha(t)| \; | u(t) - h_0(t) |$$

Thus

$$\sum_{t\in A} | \alpha(t)[u(t) - h_0(t)]| = \parallel u - h_0 \parallel = \sum_{t\in A} \alpha(t)[u(t) - h_0(t)] ,$$

and so $\alpha(t)[u(t) - h_0(t)] \geq 0$, for $t \in A$. On the other hand, $|\alpha(t)| > 0$, $\sum_t \in A|\alpha(t)| = 1$ and

$$|u(t) - h_0(t)| \leq \parallel u - h_0 \parallel \quad , \quad \text{for} \quad t \in A .$$

It follows that $|u(t) - h_0(t)| = \parallel u - h_0 \parallel$, for all $t \in A$, that is, $A \subset \text{crit}(u - h_0)$.

since u not in B then $u(t) - h_0(t) \neq 0$, for $t \in A$. So $\alpha(t) \neq 0$, for $t \in A$ implies that
$\alpha(t)[u(t) - h_0(t)] \neq 0$, for all $\in A$. Therefore,

$$\alpha(t)[u(t) - h_0(t)] > 0 \quad , \quad \text{for all} \quad t \in A,$$

that is, $\sigma(t) = \text{sgn}\, \alpha(t) = \text{sgn}\, [u(t) - h_0(t)]$, for all t in A . Since $A \subset\text{crit}(u - h_0)$ then $u(t) - h_0(t) = \sigma(t)\|u - h_0\|$, for $t \in A$. Thus $(4) \rightarrow (5)$. Also, obviously $(5) \rightarrow (4)$. ■

Remark 1. In Theorem 7. (4), (5), $1\leq r \leq n + 1$ In the actual instance and $1 \leq r \leq 2n + 1$ in the complex case and the set A is a basic set for B and u.

Theorem 8: If r is the smallest integer such that part (4) of Theorem 7 is satisfied then for each $j = 1, \dots, r$ the functional $v(t_1), \dots, \widehat{v(t_j)}, \dots, v(t_r)$ are linearly independent.

Proof: Let there exists $j, 1 \leq j \leq r$ such that the functionals

$$v(t_1), \dots, \widehat{v(t_j)}, \dots, v(t_r) ,$$

are linearly dependent. So $v(t_k) \in \text{sp}\{v(t_i): 1 \leq i \leq r, i \neq j, k\}$ for some $k \neq j$ $v(t_j) \in \text{sp}\{v(t_i): i \neq j, 1 \leq i \leq r\}$ then $(t_j), v(t_k) \in \sup\{ v(t_i):\; i \neq j\, ,k\, ,\; 1 \leq i \leq r \}$, that is, $\dim(\text{sp}\,\{ v(t_i) :\; 1 \leq i \leq r \}) \leq r - 2$.

Therefore $\dim(\text{B}|_A) \leq r - 2$ *since* ($\sup\{v(t_i): 1 \leq i \leq r\} = (\text{B}|_A)^*$ and $\dim \text{B}^* = \dim \text{B}$). Let $h_0 \in P_\text{B}(u)$. It follows that $0 \in \text{co}\{(u(t) - h_0(t))v(t): t \in A\}$. Now by implication $(1) \rightarrow (4)$ in Theorem 7, applied to $\text{B}|_A \subseteq C(A), u|_A$ and $h_0|A$, there exists a subset $A' \subseteq A$ with card $A' \leq \dim\text{B}|_A + 1 \leq r - 1$ and there exist non-zero $\alpha'(t)$ for each $t \in A'$ with $\sum_{t\in A'} |\alpha'(t)| = 1$, such that

$$\sum_{t\in A'} \alpha'(t)\, v(t) \;=\; 0 \,.$$

But by our hypothesis r is the smallest integer such that $\sum_{t\in A} \alpha(t)v(t) = 0$ is satisfied. Which is a contradiction. This completes the proof. ■

Remark 2. In the proof of above Lemma, we claimed that sp $e(\mathcal{T}) = \text{B}^*$. If it is not, then sp $e(\mathcal{T}) \subsetneq \text{B}^*$.Then there exists $\varphi \in \text{B}^*\backslash\{0\}$ such that $\varphi(\text{sp}\; v(\mathcal{T})) = \{0\}$. But $\varphi = h$ for some $h \in \text{B}$ and $v(\mathcal{T})(h) = \{0\}$, that is, $h(\mathcal{T}) = \{0\}$, which is a contradiction. Finally, future research may focus on translating these findings into real-world scenarios, such as mathematical optimization as in [10].

3. CONCLUSIONS

According to Kolmogorov's theorem, the best approximation in a real standard linear space X is described. Furthermore, the concepts of proximal set, smooth space, sun, sun point, and their relationship with the Kolmogorov condition are discussed. The effectiveness of using the best approximation in situations where high accuracy in calculating standard linear spaces is required is revealed.

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Arabic Abstract

إن التقريب الأمثل للمجموعات الفرعية للدوال داخل الفضاءات القياسية يسهل نمذجة البيانات وإدارة الأنظمة الخطية وغير الخطية. في هذه البحث، يتم وصف أفضل تقريب في فضاء خطي قياس حقيقي X بواسطة نظرية كولموغوروف. بالإضافة إلى ذلك، تتم مناقشة مفاهيم المجموعة القريبة، والفضاء السلس، والشمس، ونقطة الشمس، وعلاقتها بشرط كولموغوروف. أخيرًا، تم تسليط الضوء على فعالية استخدام أفضل تقريب في المواقف العملية لتحقيق دقة عالية في حساب الفضاءات الخطية القياسية.