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Results on Almost Nonexpansive Mappings in 2-Normed Spaces

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A B S T R A C T

The spaces of 2-normed and quasi 2-normed have been called to achieve the purpose of this article. Here, a novel mappings class is presented as non-expansive in quasi2-normed spaces. The properties of mappings class are elaborated in this study. The results of fixed point mappings are proved.

1. INTRODUCTION

Gähler (see, [1]) constructed the concepts of two (normed and Banach) spaces theories. These concepts are derived from the idea of area in 2-dimensional Euclidean space [2-3]. The theory of fixed point is crucial subject in developing functional investigation. Besides, it's utilized significantly in several branches sciences. So have subsequently been studied by many mathematicians in these spaces. Recently, Rumlawang [4] derived a norm from the 2-norm to demonstrate the theory of fixed point which is provided studying approach and the sequences of Cauchy, as well as, the contractional of mappings in the spaces of 2-normed. Anjum and Abbas [5] proved partially the theorems of fixed point as extending the outcomes by Berinde and Pacurar [6] in terms of 2-norm spaces. Harikrishnan and al.et [7] presented the two kind convergence (strong and weak) in 2-probabilistic normed spaces to be strongly or weakly bounded. The branch of fixed point theory includes a wide area of work that can be studied in current spaces as in [8-9] and its references. In this paper, some requirements are presented to prove fixed point results in quasi 2-normed spaces.

Definition 1.1: [1] assume H is dimension in a real linear space > 1 and $\|.,.\|: H\times H \rightarrow R$ satisfying the following:–

$(2N_1)$ $\|n,m\| = 0 \Leftrightarrow n$ *and* m are linearly dependent in H

$(2N_2)$ $\|n,m\| = \|m,n\|$, for all $n,m \in H$

$(2N_3)$ $\|n, \alpha m\| = |\alpha|\,\|n,m\|$ for every real number α

$(2N_4)$ $\|n+m,h\| \leq \|n,h\| + \|m,h\|$ for all $n,m,w \in H$

Then the twosome $(H, \|.,.\|)$ is named 2-norm linear space.

Herein, a 2– norm $\|n,m\|$ is good enough $\|m+\alpha n, n\| = \|m,n\|$ for all $n,m \in H$ and all scalars α

Example 1.2 : [2] Let $W = R^3$. Define $\|n,m\| = max\{|n_1 m_2 - n_2 m_1|, |n_1 m_3 - n_3 m_1|, |n_2 m_3 - n_3 m_2|\}$, where $n = (n_1, n_2, n_3)$, $m = (m_1, m_2, m_3) \in R^3$. Then $\|n,m\|$ is 2-norm on R^3

Example 1.3 : [2] Let P_n = the polynomials of h to the value $\leq n$, on a range of $[0,1]$. As known, P_n is a linear space. Let $\{n_1, n_2, \ldots\ldots\ldots n_{2k}\}$ be a dependent in fixed points of $[0,1]$ and the P_n represented by 2-normd in $\|h,g\| = \sum_i^{2k} |h(n_k)\, g'(n_k) - h'(n_k)\, g(n_k)| \rightarrow (P_n, \|h,g\|)$ is a 2-normed space (see[1]).

Park [9] presented the idea of space in the quasi with 2-norm as the following

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Definition 1.4: [10] A function on$H \times H \to R$ is a quasi – 2– normed if satisfies conditions $(2N_1), (2N_2), (2N_3)$ of definition(1.1) and the conditions $(2N_4): \exists\ K \geq 1$ such that $\|n+m, w\| \leq k\ \|n, w\| + k\ \|m, w\|$ for all $n, m, w \in H$
The couple of $(H, \|.,.\|)$ is defined by quasi two– norm space (shortly q_2– normed space).
The q2-normed $\|.,.\|$ represent the quasi- $(2;\rho)$ - normed
The collection of all 2-normed spaces are subclass of q_2-normed spaces (for $k=1$) I
$(0<p\leq 1)$ *if* $\|n+m, w\|^p \leq \|n, w\|^p + \|m, w\|^p$ *for all* $n, m, w \in H$.
n the next paragraph we recall (with details) examples of the q_2-norm spaces aren't the 2- norm spaces.

Example 1.5: [3] let $H=R^3$ and
$u=u_1 i + u_2 j + u_3 k,\ v=v_1 i + v_2 j + v_3 k \in R^3$.
Define

$$\|u, v\| = k\left|u_{i0}\, v_{i0} + 1 - u_{i0+1}\, v_{i0}\right| + \sum_{i \neq i_0}^{3} \left|u_i\, v_{i+1} - u_{i+1}\, v_i\right|$$

Where

$$\left|\mathbf{u}_{i0}\, \mathbf{v}_{i0} + 1 - \mathbf{u}_{i0} + \mathbf{v}_{i0}\right| = \min\left\{\left|\mathbf{u}_i\, \mathbf{v}_{i+1} - \mathbf{u}_{i+1}\, \mathbf{v}_i\right| : 1 \leq i \leq 3\right\}$$

$u_4 = u_1, v_4 = v_1$ *and* $k>1$.Then $(R^3, \|u, v\|)$ is a q2-normed spaces .
Is not a 2-normed space.
For $u=(0,1,-1)$; $v=(0,1,2)$ and $w=(1,0,0)$ we have
$\|u, v+w\| = \|(0,1,-1), (1,2,1)\| = k.1+3+1 = k+4$
$\|u, v\| = \|(0,1,-1), (0,2,1)\| = k.0+3+0 = 3$
$\|u, w\| = \|(0,1,-1), (1,0,0)\| = 1+k.0+1 = 2$
And $\|u, v+w\| = k+4 > \|u, v\| + \|u, w\| = 3+2 = 5$.
In this case the $(2N_4)$ will not complete the satisfaction.
Accordingly, at each $k>1$, the q2-norm spaces $(R^3, \|u, v\|)$ will not able represent the space of two-norm.

<u>Example 1.6</u>: [3] suppose P_2 refer to the two degree of real multinomial, b the range of $[0,1]$. Bearing in mind the adding and vector multiplication, P_2 represent the vector space linearly. Suppose that $\{u_1, u_2, u_3, u_4\}$ is fixed points in the interval of $[0,1]$. Described by the q2-normed on P_2 in the following

$$\|J, M\| = k\left|f(u_{i0})M`(u_{i0}) - f`(u_{i0})M(u_{i0})\right| + \sum_{i \neq i_0}^{4}\left|J(u_i)M`(u_i) - J`(u_i)M(u_i)\right|,$$

Where

$$\left|J(u_{i0})M`(u_{i0}) - J`(u_{i0})M(u_{i0})\right| = \min\left\{\left|J(u_i)M`(u_i) - J`(u_i)M(u_i)\right| : 1 \leq i \leq 4\right\}$$

and $k>1$.
Then $(P_2, \|J, M\|)$ is a q2-normed spaces.
At last, let us show that $(P_2, \|J, M\|)$ defined as above, is not a 2-normed space.
Let us consider the case $u_1=1, u_2=\frac{1}{2}, u_3=\frac{1}{3}$ and $u_4=0=$ For $J=u,\ g=u^2$
And $L=(u-1)$, we have

$$\left|J(u_1)(M+L)`(u_1) - J`(u_1)(M+L)(u_1)\right| = |2.0 - 2.1| = 2$$

$$\left|J(u_2)(M+L)`(u_2) - J`(u_2)(M+L)(u_2)\right| = \left|\frac{5}{4}\ .\ (-1) - 1\ .\ \frac{5}{4}\right| = \frac{5}{2}$$

$$\left|J(u_3)(M+L)`(u_3) - J`(u_3)(M+L)(u_3)\right| = \left|\frac{10}{9}\ .\left(-\frac{4}{3}\right) - \frac{2}{3}.\frac{13}{9}\right| = \frac{66}{27} = \frac{22}{9}$$

$$\left|J(u_4)(M+L)`(u_4) - J`(u_4)(M+L)(u_4)\right| = |1.(-2) - 0\ .\ 2| = 2$$

And

$$\|J, M+L\| = k.2 + \frac{5}{2} + \frac{22}{9} + 2 = k.2 + \frac{125}{18}\ .$$

Also, we have

$$\left|J(u_1)M`(u_1) - J`(u_1)M(u_1)\right| = |2.0 - 2\ .\ 1| = 2$$

$$\left|J(u_2)M`(u_2) - J`(u_2)M(u_2)\right| = \left|\frac{5}{4}\ .\ (-1) - 1\ .\ \frac{1}{4}\right| = \frac{3}{2}$$

$$\left|J(u_3)M`(u_3) - J`(u_3)M(u_3)\right| = \left|\frac{10}{9}.\left(-\frac{4}{3}\right) - \frac{2}{3}\ .\ \frac{4}{9}\right| = \frac{48}{27} = \frac{16}{9}$$

$$\left|J(u_4)M`(u_4) - J`(u_4)M(u_4)\right| = |1.(-2) - 0\ .\ 1| = 2$$

And $\|J, M\| = k\ .\ 0 + \frac{3}{2} + \frac{16}{9} + 2 = \frac{95}{18}$
Also, we have

$\left|J(u_1)M`(u_1) - J`(u_1)M(u_1)\right|$
$= \left|2.0 - 2\ .\ 1\right| = 2$
$\left|J(u_2)M`(u_2) - J`(u_2)M(u_2)\right| = \left|\frac{5}{4}\ .\ 0 - 1.\ 1\right| = 1$

$\left|J(u_3)M`(u_3) - J`(u_3)M(u_3)\right| = \left|\frac{10}{9}.0 - \frac{2}{3}.\ 1\right| = \frac{2}{3}$

And $\left|J(u_4)M`(u_4) - J`(u_4)M(u_4)\right| = \left|1\ .\ 0 - 0.\ 1\right| = 0$

$\|J, L\| = 2 + 1 + \frac{2}{3} + k.0 = \frac{11}{3}$. From the above results, we get :

$\|J, M + L\| = k.2 + \frac{125}{18} > \|J, M\| + \|J, L\| = \frac{95}{18} + \frac{11}{3} = \frac{161}{18}$.

Therefore, for every $k > 1$, the q2– normed space ($P_2, \|J, M\|$) is not a 2-normed space.

Definition 1.7: [11] A succession $\{t_n\}$ within the q2–norm space (H, $\|.,.\|$) can be a Cauchy sequence if $\lim_{m,n\to\infty} \|t_n - t_m, w\| = 0$ for each w instead of H.

Definition 1.8: [11] An order of $\{t_n\}$ into the q2– norm space (H, $\|.,.\|$) can be similar if n point is included in H such that $\lim_{n\to\infty} \|t_n - t, w\| = 0$ for all w in H.
Consideing $\{t_n\}$ is similar to t, we write $\{t_n\} \to t$ as $n \to \infty$.

Definition 1.9: [11] Considering each Cauchy sequence is approching the element H. Then the space with linear q2–norm (H, $\|.,.\|$) can be completed.

Definition 1.10: [11] A completed q2 with respect to it's norm space can be defined as q2-vector space (Banach).

Definition 1.11: Mapping $T{:}H \to H$ be a suppose where (H, $\|.,.\|$) is a q2– normed space. For all $h \in H$, let $O(h) = \{h, Th, T^2h, \ldots\}$ can represent the orbit of Tat h. (H, $\|.,.\|$) is defiend by T-orbitally complete in only the case of each Cauchy sequence is enclosed by $O(h)$ collect to the H point.

Roshan elat. [12] proved some lemmas about sequences in a general metric space, here, we reform and prove in q2–normed space .

Lemma 1.12: Permit (H, $\|.,.\|$) be a q2– normed including the constant $k \geq 1$ and $\{h_u\}$ is the approximated sequence in H with $\lim_{u\to\infty} h_u \to h$. Then for all $u \in H$

$$s^{-1}\|h - w, T\| \leq \liminf_{u\to\infty} \|h_u - w, T\|$$
$$\leq \limsup_{u\to\infty} \|h_u - w, T\| \leq s\|h - w, T\|$$

Proof : It is clear by definitions of limit and q2-normed function .
From lemma(1.12), it follows that if $\{h_u\} \subseteq H$ is a sequence such that $\lim_{u\to\infty} h_u \to h$ for some $h \in H$, then $\lim_{n\to\infty} \|h_n - h, T\| = 0$.

Lemma 1.13: Let $\{H_n\}$ be a sequence in a q2–normed (H, $\|.,.\|$) such that $\lim_{u\to\infty} \|h_n - h_{n+1}, T\| = 0$
If $\{h_n\}$ isn't the sequence of Cauchy, so the existence of $\varepsilon > 0$ and the 2-sequences $\{m(k)\}$ and $\{n(k)\}$ of positive numbers as following

$$\varepsilon \leq \liminf_{k\to\infty} \|h_{m(k)} - h_{n(k)}, T\|$$
$$\leq \limsup_{k\to\infty} \|h_{m(k)} - h_{n(k)}, T\| \leq s\varepsilon,$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \|h_{m(k)} - h_{n(k)+1}, T\|$$
$$\leq \limsup_{k\to\infty} \|h_{m(k)} - h_{n(k)+1}, T\| \leq s^2\varepsilon,$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \|h_{m(k)+1} - h_{n(k)}, T\|$$
$$\leq \limsup_{k\to\infty} \|h_{m(k)+1} - h_{n(k)}, T\| \leq s^2\varepsilon,$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \|h_{m(k)+1} - h_{n(k)+1}, T\|$$
$$\leq \limsup_{k\to\infty} \|h_{m(k)+1} - h_{n(k)+1}, T\| \leq s^3\varepsilon\ .$$

Proof. If $\{h_n\}$ isn't classified as a Cauchy sequence, so the presence of $\varepsilon > 0$ and the sequences $\{m(k)\}$ and $\{n(k)\}$ of positive numbers as following

$$n(k) > m(k) > k,\quad \|h_{m(k)} - h_{n(k)-1}, T\| < \varepsilon,\quad \|h_{m(k)} - h_{n(k)}, T\| \geq \varepsilon$$

for all positive integers k. Now, using the triangle inequality we have

$\varepsilon \leq \left\| h_{m(k)} - h_{n(k)}, T \right\| \leq s$
$\left[\left\| h_{m(k)} - h_{n(k)-1}, T \right\| + \left\| h_{n(k)-1}, h_{n(k)}, T \right\| \right]$
$< s\varepsilon + s \left\| h_{n(k)-1}, h_{n(k)}, T \right\|$.

Taking the upper and lower limits as $k \to \infty$, $\varepsilon \leq \lim\inf_{k\to\infty} \left\| h_{m(k)}, h_{n(k)}, T \right\|$

$\leq \lim\sup_{k\to\infty} \left\| h_{m(k)}, h_{n(k)}, T \right\| \leq s\varepsilon$.

Using the triangle inequality again we have

$\left\| h_{m(k)} - h_{n(k)}, T \right\| \leq s$
$\left[\left\| h_{m(k)} - h_{n(k)+1}, T \right\| + \left\| h_{n(k)+1} - h_{n(k)}, T \right\| \right]$
$\leq s^2 \left[\left\| h_{m(k)} - h_{n(k)}, T \right\| + \left\| h_{n(k)+1} - h_{n(k)}, T \right\| \right]$
$+ s \left\| h_{n(k)+1} - h_{n(k)}, T \right\|$

Taking the upper and lower limits as $k \to \infty$,

we have

$$\varepsilon \leq s \lim\sup_{k\to\infty} \left\| h_{m(k)} - h_{n(k)+1}, T \right\|$$
$$\leq s^3 \varepsilon ,$$

Or, equivalently,

$$\frac{\varepsilon}{2} \leq s \lim\sup_{k\to\infty} \left\| h_{m(k)} - h_{n(k)+1}, T \right\|$$
$$\leq s^2 \varepsilon .$$

Definition 1.14 : Assume $\varphi : [0, \infty) \to [0, \infty)$ can be distance function as subadditive variation if

(i) φ is the variation of distance function (i.e., φ is continuous, strictly growing and $\varphi(t) = 0$ if and only if $t = 0$) ,

(ii) $\varphi(u+m) \leq \varphi(u) + \varphi(m)$ $\forall\ u, m \in [0, \infty)$

As mentioned in [12] , $\varphi_1(u) = ku$ for some $k \geq 1, \varphi_2(u) = \sqrt[n]{u}$, and $n \in N, \varphi_3(u) = \log(1+u)$, $u \geq 0$
$\varphi_4(u) = \tan^{-1} u$

are some examples of sub − additive altering distance functions .

Definition 1.15: suppose that $(H, \|.,.\|)$ is the q_2– norm space. The replacement of $T: H \to H$ can relativly be not scalable mapping on H if positive real number is presented $p < 1$ as in the following $u, v \in H$, $s^q \varphi(\|Tu - Tv, T\|)$

$\leq p\, Q(u, v) + \varphi(\|u - v, T\|)$

For some $q \geq 5$, where $Q(u, v) = |\varphi(\|Tu - v, T\|) - \varphi(\|u - Tv, T\|)|$

$or\ \varphi(\|u - Tu, T\|) + \varphi(\|v - Tv, T\|)$

Example 1.16: From the definition, each not extensive mapping on a q_2-normed space $(H, \|.,.\|)$ is nearly not extensive mapping.

Consider q_2–normed space and $C \subseteq H$, if $C \subseteq H$, for all $u \in C$ and $h \in T$, the range of constant points of T.

Relatively nonexpansive representing with $Q(u, v) =$

$|\varphi(\|Tu - v, T\|) - \varphi(\|u - Tv, T\|)|$

is a quasi – nonexpansive mapping. For,it $h \in T$ then $\varphi(\|Tu - Th, T\|)$

$\leq s^q \varphi(\|Tu - Th, T\|)$

$\leq p|\varphi(\|Tu - h, T\|) - \varphi(\|u - Th, T\|)|$

$+ \varphi(\|u - h, T\|)$

$= p\{\varphi(\|Tu - Th, T\|) - \varphi(\|u - Th, T\|)\}$

$+ \varphi(\|u - h, T\|)$, if $\|Tu - h, T\| >$

$\|u - Th, T\|$ $p\{\varphi(\|u - Th, T\|) - \varphi(\|Tu - h, T\|)\}$

$+ \varphi(\|u - h, T\|)$, if $\|Tu - h, T\| > \|u - Th, T\|$,

Acoordingly, in each of those conditions, the we got $\varphi(\|Tu - Th, T\|) \leq \varphi(\|u - h, T\|)$,

that is, $\|Tu - h, T\| \leq \|u - h, T\|$. However, with $Q(u, m) = \varphi(\|u - Tu, T\| + \varphi(\|m - Tm, T\|))$, May be the relatively non-expansive mapping is not a q-non-expansive mapping.

Theorem 1.17 : Let $(H, \|.,.\|)$ $be T$- orbitally complete q_2– normed with $k = 1$ or $k \geq 2$ and $T : H \to H$ be an almost nonexpansive mapping with

$\|T_n - T_m, w\|$

$\leq K\|n, w\| + K\|m, w\|$

For some $q \geq 5$ and $0 \leq p < 1$ $with$ $kp < 1$. If T is asymptotically regular at $h_0 \in H$, then T has a fixed point .

Proof : Since T is is asymptotically regular at $h_0 \in H$ we have $\lim_{n\to\infty} \|T^n h_0, T^{n+1} h_0, T\| = 0$

Firstly, we prove that $\{h_n\} \subseteq H$ defined by $h_n = T^n h_0$ represent Cauchy chain where, each $\varepsilon > 0$ one able to find $k \in W$ within $\forall\, h \geq k$, $\left\| h_t , h_g, T \right\| < \varepsilon$.

Otherwise, Lemma (1.13) existence of $\varepsilon > 0$ where the the subsequence was able to be founded $\{H_{h(i)}\}$ and $\{H_{t(i)}\}$ of $\{H_n\}$ with $h(i) > f(i) \geq i$ and

(a) $f(i) = 2t'$ and $h(i) = 2t + 1$, where t and t are nonnegative integer,

(b) $\rho\left(H_{f(i)}, H_{h(i)}\right) \geq \varepsilon$, and

It's the lowest number in this case (b) (c) $h(i)$.

believes is that , $\rho\left(H_{t(i)}, H_{g(i)-1}\right) < \varepsilon$

Hence by continuousness, sub– uniformity, and rising the characteristics of φ we got

$$\varphi(\varepsilon) \leq \left(\limsup_{i\to\infty} k \left\| h_{t(i)} - h_{g(i)}, T \right\|\right),$$

$$\leq S\,\varphi(\varepsilon),$$

$$\varphi(\varepsilon) \leq S\,\varphi(\varepsilon),$$

$$\frac{\varphi(\varepsilon)}{s} \leq \varphi\left(\limsup_{i\to\infty} k \left\| h_{t(i)} - h_{g(i)}, T \right\|\right)$$

$$\leq S^2 \varphi(\varepsilon),$$

$$\frac{\varphi(\varepsilon)}{s} \leq \varphi\left(\limsup_{i\to\infty} k \left\| h_{t(i)+1} - h_{g(i)}, T \right\|\right)$$

$$\leq S^2 \varphi(\varepsilon),$$

$$\frac{\varphi(\varepsilon)}{s^2} \leq \varphi\left(\limsup_{i\to\infty} k \left\| h_{t(i)+1} - h_{g(i)+1}, T \right\|\right)$$

$$\leq S^3 \varphi(\varepsilon).$$

From (1.15), we have,

$$S^q \varphi\left(k \left\| h_{t(i)+1} - h_{g(i)+1}, T \right\|\right)$$

$$\leq k \left| \varphi\left(\left\| h_{t(i)} - h_{g(i)}, T \right\|\right) - \varphi\left(\left\| h_{t(i)} - h_{g(i)+1}, T \right\|\right) \right|$$

$$+ \varphi\left(\left\| h_{t(i)} - h_{h(i)}, T \right\|\right).$$

Taking the upper limit as $i \to \infty$, we get

$$S^q \varphi\left(\limsup_{i\to\infty} \left\| h_{f(i)+1} - h_{h(i)+1}, T \right\|\right)$$

$$\leq \rho \left| S^2 \varphi(\varepsilon) - \frac{\varphi(\varepsilon)}{s} \right| +$$

$$s\varphi(\varepsilon) < 2S^2 \varphi(\varepsilon),$$

Or,

$$\varphi\left(\limsup_{i\to\infty} \left\| h_{m(i)+1} - h_{h(i)+1}, T \right\|\right)$$

$$< \frac{2\varphi(\varepsilon)}{s^{q-2}} < \frac{2\varphi(\varepsilon)}{s^3}$$

Since $q \geq 5$, contradicting for $k \geq 2$. We note here that if $k = 1$, then T is approximately unvarying at $h_0 \in H$ indicates $\{h_n\}$ represent Cauchy sequence, that simply observed. So $\{g^n h_0\}$ is also Cauchy sequence and subsequently $(H, \|.,.\|)$ is completely orbital, and it exists $h \in H$ for example $\lim_{h\to\infty} g^n h_0 = Z$.

Now, applying (1.15)

$$\varphi(\|z - gz, T\|) \leq s\varphi\left(\left\| h - g^{n+1} h_0, T \right\|\right)$$

$$+ s^q \varphi\left(\left\| g^{n+1} h_0 - gh, h \right\|\right)$$

$$\leq s\varphi\left(\left\| h - g^{n+1} h_0, T \right\|\right) +$$

$$\left| \varphi\left(\left\| h - g^{n+1} h_0, T \right\|\right) - \varphi\left(\left\| g^n h_0 - gh, T \right\|\right) \right|$$

$$+ \varphi\left(\left\| g^n h_0 - h, T \right\|\right)$$

Considering this limits $n \to \infty$, by lemma (1.13)

The outcomes, $(1 - s\rho)\ \varphi(\|h - gh, T\|) \leq 0$,

Hence, $\varphi(\|h - gh, T\|) = 0$ or $\|h - gh, T\|$

i.e., $gh = h$,

as required.

Remark 1.18: In theorem (1.17), T got an exclusive constant point in case of $(H, \|.,.\|)$ isn't a quasi-2-normed (i.e,$k \neq 1$). Where, if $w \in Y$ is also a fixed point of T,

Therefore, from (1.15)

$$s^q \|h - w, T\| \leq p\, \big| \|h - w, T\| - \|h - w, T\| \big|$$

$$+ \|h - w, T\|,$$

Which implies $(h - w, T) = 0$ (since $k \geq 2$), as claimed.

Corollary 1.19: A nonexpansive mapping on a q_2– normed $(H, \|.,.\|)$ got a fixed point in case of asymptotically unvarying at any point of H .

Proof. The proof is followed based on Theorem (1.17) by considering φ employing the identified mapping $p=0$ *and* $k=1$.

Theorem 1.20: Let $(H, \|.,.\|)$ be a complete q_2– normed and $T: H \to H$ be a mapping such that for all $u, m \in H$, $s^q \varphi(\|Tu - Tm, T\|)$

$$\leq p\{\varphi(\|u - Tu, T\|) + \varphi(\|m - Tm, T\|)\}$$

$$+ \varphi(\|u - m, T\|),$$

In few non-negative real number $p < 1$ with $kp < 1$ and $q \geq 5$.

In case of T is representing asymptotically stable at a point u_0 *of* H, then T have a fixed point.

Proof. as T is approximately stable at $u_0 \in H$, $\lim_{n \to \infty} \|T^n u_0 - T^{n+1} u_0, T\| = 0$.

Thus the sequence $\{u_n\}$ defined by $u_n = T^n u_0$, $n \in W$ is a Cauchy sequence as proven in Theorem (1.17). As $(H, \|.,.\|)$ is integral, so it exists $h \in H$

For example

$$\lim_{n \to \infty} u_n = h.$$

Herein, meanwhile φ is sub-additive and sub-identical varying distance function and

$$q \geq 5, \quad \varphi(\|h - Th, T\|)$$

$$\leq s\varphi\left(\|h - T^{n+1}u_0, T\|\right) +$$

$$s\varphi\left(\|T^{n+1}u_0 - Th, T\|\right)$$

$$\leq s\varphi\left(\|h - T^{n+1}u_0, T\|\right) +$$

$$s^q\varphi\left(\|T^{n+1}u_0 - Th, T\|\right).$$

In the above expression, we get,

$$\varphi(\|h - Th, T\|)$$

$$\leq s\varphi\left(\|h - T^{n+1}u_0, T\|\right)$$

$$+ p\left\{\varphi\left(\|T^n u_0 - T^{n+1}u_0, T\|\right) + \varphi(\|h - Th, T\|)\right\}$$

$$+ \varphi\left(\|T^n u_0 - h, T\|\right).$$

Considering the range of $n \to \infty$ and utilizing Lemma (1.13), the result, $(1 - sp)\ \varphi(\|h - Th, T\|) \leq 0$,

That demonstrates $Th = h$, as required.

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Arabic Abstract

لغرض تحقيق هدف هذه المقالة تم اعادة تعريف كل من الفضاءات 2 - المعيارية و شبه 2- معيارية. هنا، نقدم فئة جديدة من التطبيقات تسمى التطبيقات اللا متمددة غالبا في الفضاءات شبه 2- معيارية. لقد تم مناقشة بعض خصائص هذه الفئة من التطبيقات. حيث تم إثبات نتائج حول النقطة الصامدة لها.