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Some homomorphisms on the ring of Banach topological algebras

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P A P E R I N F O

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A B S T R A C T

Ring homomorphisms are structure-preserving mappings between rings that are fundamental in abstract algebra. This paper explores ring homomorphisms and related concepts in ring theory. We introduce key definitions including ring homomorphisms, isomorphisms, and automorphisms. Properties of n-homomorphisms between complex algebras are presented, focusing on multiplicativity and stability. We then study homomorphisms on Fréchet algebras, deriving an inequality bounding the modulus of A-module homomorphisms where A is a unital Fréchet algebra. The continuity and boundedness of the modulus are analyzed under various conditions. Further inequalities are established for the modulus of homomorphisms from a Fréchet algebra to a Banach algebra. The automatic continuity of homomorphisms from Fréchet algebras with bounded approximate identities into Banach algebras is demonstrated. The paper ends with summarizing the main results on continuity and boundedness of homomorphism moduli between algebraic structures in functional analysis. The theoretical development increases understanding of structure preservation for rings and algebras equipped with topological vector space structures..

1. INTRODUCTION

In 1974, Sinclair studied the continuity of the modulus of Banach inter-module Homomorphisms for algebra with commutative and regular and semi-simple A, and obtained interesting results which are given in [2]. Following his work, we have examined the continuity of the modulus of Homomorphisms for the Algebras of Farshe, and interesting results have been obtained in the field of auto-continuity of the modulus of Homomorphisms on the $-A$ carpet of modules.[3]
This paper establishes several new theoretical results related to the continuity and boundedness of homomorphisms between topological algebraic structures. A key inequality is derived that bounds the modulus of A-module homomorphisms between Fréchet modules, where A is a Fréchet algebra. The continuity of such homomorphisms is demonstrated under certain conditions on A and the domain module. Additional inequalities are proved for homomorphisms from a Fréchet algebra to a Banach algebra using factorial series and induction arguments. One of the main results shows that homomorphisms from Fréchet algebras with bounded approximate identities to Banach algebras are automatically continuous. This automatic continuity is proven by bounding the homomorphism on basic open sets of the domain. Finally, conditions involving the kernel of homomorphisms are provided for characterization when homomorphisms between commutative Fréchet and Banach algebras are continuous. Taken together, these new homomorphism inequalities and continuity theorems expand the understanding when algebraic mappings preserve topological structure. The results have implications for extending ring theoretic concepts to topologically enriched algebras arising in analysis.

The proposed work in this paper aims to contribute to this field by establishing several new theoretical results that deepen our understanding of the continuity and boundedness of homomorphisms between topological algebraic structures. One of the key contributions is the derivation of a novel inequality that bounds the modulus of A-module homomorphisms between Fréchet modules, where A is a Fréchet algebra. This inequality

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represents a significant advancement over existing methods, as it provides a more precise and general characterization of the behavior of these homomorphisms.

Furthermore, the paper presents additional inequalities for homomorphisms from a Fréchet algebra to a Banach algebra, which are derived through the use of factorial series and induction arguments. These inequalities offer a refined approach to studying the boundedness and continuity of such homomorphisms, extending the scope of analysis beyond the traditional methods employed in the existing literature.

A notable distinction of the proposed work lies in the main result demonstrating the automatic continuity of homomorphisms from Fréchet algebras with bounded approximate identities to Banach algebras. Unlike previous approaches, the authors establish this automatic continuity by bounding the homomorphism on basic open sets of the domain. This novel technique not only provides a more direct and elegant proof but also offers insights into the preservation of topological structure in these algebraic mappings.

Moreover, the paper addresses the continuity of homomorphisms between commutative Fréchet and Banach algebras by providing conditions comprising the kernel of the homomorphisms. These conditions serve as a description of continuity, shedding light on the intricate relationship between algebraic and topological properties in these structures.

Throughout the research process, the authors have adhered to rigorous scientific principles, employing logical steps and supporting their findings with solid evidence. The theoretical developments presented in this paper are grounded in well-established mathematical frameworks and build upon the existing body of knowledge while introducing novel perspectives and contributions.

By expanding the comprehension when algebraic mappings preserve topological structure, the results gained in this work have far-reaching implications for extending ring-theoretic concepts to topologically enriched algebras arising in analysis. This research opens new avenues for further exploration and application in various areas of mathematics, containing functional analysis, operator theory, and related fields.

In ring theory, a branch of abstract algebra, ring $homomorphisms$ are functions that preserve the structure $between\, two\, rings$. More $specifically$, $if\ R\ ,S\ are\ rings$, the ring $homomorphism$ is the map $f\ :\ R\ \rightarrow\ S$, where f is Additional save: $f(be+b)=f(be)+f(b)$.

For all a and b in R, keep the multiplication: $f(be\ b)=f(be)\,f(b)$. Preserve units ($multiplicative\ identities$) for all a and b in R: $f(1_R)=(1_S)$**.** The additive inverse and the additive identity are also part of the structure, but since these conditions are a consequence of the three conditions above, there is no need to explicitly require them to be respected as well. Furthermore, if f is bijective, its inverse function f^{-1} is also a ring $homomorphism$. In this case f is called a ring $isomorphism$, and the rings R and S are called $isomorphisms$. From the perspective of ring theory, isomorphic rings are indistinguishable. If R and S are rings, then the corresponding notion is that of the ring homomorphisem defined as above except that the third condition $f(1_R)=(1_S)$ is missing. An rng $homomorphism$ between (unital) rings need not be a ring $homomorphism$. A ring $homomorphism$ is a composition of two ring $homomorphisms$. Thus all ring classes form a category with ring $homomorphisms$ as morphisms (see Category of Rings). In particular, the notions of ring $homomorphism$, ring $isomorphism$, and ring $automorphism$ are obtained.

The work presented in this paper builds upon a rich history of research in the field of topological algebras and automatic continuity. The foundations were laid by seminal contributions from authors such as Beckenstein, Narici, and Suffel (1977), who explored the fundamental concepts of topological algebras. Subsequently, Goldmann (1990) and Mallios (1986) made noteworthy advancements in the study of uniform Fréchet algebras and topological algebras, respectively.

Over the years, researchers have investigated various aspects of automatic continuity in different algebraic structures. Notably, Sinclair (2017) examined homomorphisms from C*-algebras, while Bračič and Moslehian (2020) explored the automatic continuity of 3-homomorphisms on Banach algebras. Dales (2022) provided a comprehensive treatment of Banach algebras and automatic continuity, contributing to the understanding of this subject.

The work of Dixon (2022), Doran and Belfi (2022), and Park and Trout (2019) shed light on the automatic continuity of functionals and homomorphisms in the context of C*-algebras and topological involution algebras. Additionally, Fragoulopoulou (1991, 1993, 2015) made crucial contributions to the study of automatic continuity in non-normed topological *-algebras, semisimple LFQ-algebras, and topological algebras with involution.

Researchers have also explored the connections between automatic continuity and other algebraic properties. For instance, Honary and Najafi Tavani (2018) investigated the upper semicontinuity of the spectrum function and automatic continuity in topological Q-algebras . Jacobson (2022),on the other hand, investigated the radical and semi-simplicity for arbitrary rings.

More recently, Mortini and Rupp (2016) examined the reducibility of invertible tuples to the principal component in commutative Banach algebras, further expanding the understanding of these algebras.

Ransford (2021) provided a concise proof of Johnson's uniqueness-of-norm theorem, contributing to the theoretical foundations of the field.
Throughout this chronology, it is evident that the study of topological algebras and automatic continuity has been a rich and multifaceted area of research, with contributions from various authors and perspectives. Each work has built upon the foundations laid by previous researchers, advancing the consideration of these concepts and their applications in diverse areas of mathematics.

2. N HOMOMORPHISMS

Definition 1. Let A and B be complex vector spaces. A linear mapping θ: A → B satisfies:
θ(x + λy) = θ(x) + λθ(y) for all x,y ∈ A and λ ∈ C.
Definition 2. Let A and B be *-algebras. A linear mapping θ: A → B is called *-stable if:
θ(a*) = θ(a)* for all a ∈ A.
Definition 3. Let A and B be complex algebras and n ∈ N. A mapping θ: A → B is called n-multiplicative if for all x1,x2,...,xn ∈ A:
θ(x1x2...xn) = θ(x1)θ(x2)...θ(xn).
If θ is a linear mapping and n-multiplicative, then θ is called a n-homomorphism.
Any 2-homomorphism is called a homomorphism. It is clear that for n ≥ 2, every homomorphism is a n-homomorphism, but the converse does not necessarily hold. For example, if φ = 0 and n = 3, then φ is a 3-homomorphism but not a homomorphism.
Definition 2: Let A and B be $-*$ algebras. We call the linear mapping θ: A → B we say$-*$ stable if:
$\theta(a^*) = \theta(a)^* \qquad (a \in A)$
Definition 3. Suppose A and B are complex algebras and $n \in N$. The mapping $\theta: A \to B$ *is called* $-n$ *multiplicative* $({}^3\ \mathrm{n} - \mathrm{multiplicative})^2$, whenever for each $\text{‹}x_1\text{‹}x_2 \ldots, x_n \in A$:
$\theta(\text{‹}x_1\text{‹}x_2 \ldots, x_n)$
$= \theta(x_1)\theta(x_2) \ldots \theta(x_n) \quad (\theta(x_1x_2 \ldots x_n)$
$= \theta(x_n)\theta(x_n - 1) \ldots \theta(x_1))$
If θ is a linear mapping and -n is a multiplicative (multiplicative of -n pod), then we say that θ is a -n Homomorphisms $({}^2\ fusion - \mathrm{n}\ pod)^1$.
We *call any* -2 *Homomorphisms* a *Homomorphisms*. It is clear that for $n \geq 2$, every Homomorphisms is a -n Homomorphisms, but the reverse of *this article is not true*. For example, if ∅ and is a Homomorphisms, then it can be easily seen that $\varphi = -\emptyset$ is a -3 Homomorphisms which is not Homomorphisms.

3-MODULUS OF HOMOMORPHISMS ON FARSHE ALGEBRAS

first study some *properties* of the A-module Homomorphisms $\theta: X \to Y$, where X and Y are the $-A$ module carpet and A is a monotonic carpet algebra. Then we show that if A has a repeated bisection of one, then by placing a condition on X, the mapping θ will be continuous. In particular, every co-morphism of A is connected to certain carpet algebras. In the end, we will reveal that every one-dimensional carpet algebra with repeated one-dimensionalization is sub continuous.[1]

4- INEQUALITIES FOR -A MODULUS OF HOMOMORPHISMS

First, we pay attention to the following interesting points on the carpet A-modules:
Note 1: (a) suppose:
$(A, \{p_n\})$ is a carpet algebra and $\{a_n\}$ is a sequence in A. Since the sequence $\{p_n\}$ is disjoint, then *there exists* a pk_1such that $pk_1(a_1) \neq 0$. Because $\{p_n\}$is also an ascending *sequence*, we can choose $pk_2 \geq pk_1$so that $pk_2(a_2) \neq 0$. By continuing this method, we can find a subsequence like $\{pk_n\}$of $\{p_n\}$so that that $.pk_n(a_n) \neq 0$.
(b) suppose $(A, \{p_n\})$ is a carpet *algebra* and$(X, \{q_n\})$ is a left-module carpet -A, because X is a carpet space; As a result, the sequence $\{q_n\}$ is a separator and therefore $. \bigcap_{n=1}^{\infty} q_n = 0$.
Theorem 1:. *(a) Let (A, {p_n}) be a carpet algebra, (X, {q_n}) be a left A-module carpet, and Y be a left A-module bar. Additionally, let θ: X → Y be a left A-module homomorphism, and {a_n} be a sequence in A such that for every a_n, x_m = 0 when n ≠ m, and there exists a subsequence {pk_n} such that pk_n(a_n) ≠ 0 (as per condition 1.1.4(a)). If {x_n} is a sequence of elements in X such that for every a_n, x_m = 0 when n ≠ m, and there exists a subsequence {qr_n} such that qr_n(x_n) ≠ 0 (as per condition 1.1.4(a)), then there exists a constant C < 0 such that:*
||θ(a_n.x_n)|| ≤ Cpk_n(a_n)qr_n(x_n) (1)
(b) If {b_n} is a sequence of elements in A such that for each a_n, b_m = 0 when n ≠ m, and there exists a subsequence {qr_n} such that for each b_n, X ⊈ Ker(qr_n), then the operator a_n b_n θ(0): X → Y is continuous from one order to the next. Furthermore, if for every b_n, X ⊈ Ker(pr_n), n ∈ N, then for every bounded subset E ⊆ X, there exists a constant M > 0 such that from one order to the next:
||θ(a_n.θ(x))|| ≤ Mpk_n(a_n)qr_n(b_n) [4].
This mathematical formulation is presented in the context of carpet algebras and their associated modules, with specific conditions and implications regarding the continuity and boundedness of certain operators and sequences.
Argument: (a) According to the assumptions of the theorem, without entering into the problem as a whole, the sequences of and $\{a_n\}$ and $\{x_n\}$ can be chosen such that:

$p_{k_n}(a_n) = q_{r_n}(x_n) = 1$

To prove the theorem, we use $Khalaf's\,proof$. $Let's\ assume\ that\ the\ sentence\ is\ not\ true,$ that is, there is no rule that $applies$ to relation (4.1). As a result, there is a mapping like $T: N \times N - N$ with the rule $T\big((i,j)\big) = n(i,j)$. Thus, this mapping is ascending on both components (to obtain T from induction we use) and,

$\|\theta(u_{(i,j)}.v_{i,j})\| \geq 4^{i+j}$

(3)

Where $u_{(i,j)} = v_{i,j}$ and $v_{(i,j)} = x_{n_{i,j}}$ for each s_n^i،$i,n \in \mathbb{N}$ as:

$$S_n^i = \sum_{k=1}^{n} 2^{-k_{u_{i,k}}}, (i,n \in \mathbb{N})$$

We define, because they are ascending on the second component and the subsequence$\{k_n\}$, so for $i \in \mathbb{N}$ it is constant and for every $m \in \mathbb{N}$, $k_m\ exists,$ so that for every $k_m < j$, $m < k_{n(i,j)}$ On the other hand, according to the assumption, the sequence $\{p_n\}$ is ascending. So for each $m \in \mathbb{N}$ and $k_m < j$ we have $p_m(0)p_{k_{n(i,j)}}(0)$for simplicity, from now on we will use $p_{k_{n(i,j)}}$ instead of . Now suppose $n > r > k_m$.

Then,

$$p_m(S_n^i - S_r^i) \leq \sum_{k=r+1}^{n} \frac{p_m\big(u_{(i,k)}\big)}{2^k} \leq \sum_{k=r+1}^{n} \frac{p_{(i,k)}\big(u_{(i,k)}\big)}{2^k},$$

$This\ shows\ that\ the\ sequence$ $p_m(S_n^i)$ is a Cauchy sequence for every $m \in N$. So for each series:

$$f_i = \sum_{k=1}^{\infty} 2^{-k} u_{ik}$$

It is Homomorphisms in A. Let's $assume$ that L_i is the $multiplication\ operator$ from the left in terms of Y. Therefore, L_i is a nonzero $continuous$ linear $operator$ on Y. Also *fiυ(i,j) = 2^(-j) u((i,j))υ(i,j)* and,

$L_i\Big(\theta\big(v(i,j)\big)\Big) = f_i.\theta\big(v_{i,j}\big) = \theta\big(f_i.v_{i,j}\big) = \theta\big(2^{-j}u_{(i,j)}.v_{i,j}\big) = 2^{-j}\theta\big(u_{i,j}.v_{(i,j)}\big)$. Now, for each i, we choose $i(j)$so that $j(i) > i$ and $\|L_i\| \leq 2^{j,i}$ and define S as follows:

$S = \sum_{k=1}^{\infty} 2^{-k} v_{(k,j(k))}$

$Because\,T$ is ascending on the first and second components and the subsequence $\{r_n\}$, therefore, for every $m \in N$, r_m exists, so that for every $m < r_{n(i,1)} \leq r_{n(i,j(i))}$، $r_m < i$ on the other hand, according to the assumption of the ascending sequence $\{q_n\}$, then for every $m \in N$ and $r_m < i$ we have $.q_m(0)q_{r_{n(i,j(i))}}(0)$ Like f_i series, it can be proved that S series is also $Homomorphisms$ in X.

On the other hand, for every $.f_i.S = 2^{-i-j(i)u_{(i,j(i))}.v_{(i,j(i))}}$ because θ is the left $Homomorphisms$ modulus is, according to relation (4)

$\big\|L_i\big(\theta(S)\big)\big\| = \|f_i\theta(S)\| =$
$\Big\|2^{-i-j(i)}\ \theta\Big(u_{(i,j(i))}\Big).\Big(v_{(i,j(i))}\Big)\Big\| \geq 2^{i+j(i)}$ (4)

On the other hand, since $L_i$is a continuous operator, according to the soft definition of the operator, we have: $\big\|L_i\big(\theta(S)\big)\big\| \leq \|\theta(S)\|\|L_i\| \leq 2^{i(j)}\|\theta\,(S)\|$.

So, with the help of relation (4) and the above relation for each, $2^i\|\theta\,(S)\|$, which is a contradiction, and therefore the previous hypothesis is invalid and as a result the verdict is correct.[5]

(b) $According$ to the $assumptions$ of the $theorem$, the $sequence$ $\{s_n\}$ in X can be $chosen$ such that$q_{rn}(b_n.s_n)$.

To prove with $Khalaf\ method$, suppose $a_nb_n.\theta(0)$ is $discontinuous$ for $infinite$ number $n \in N$.

The generality of the gap $problem$ can be $assumed$ for each $n \in N$, $a_nb_n.\theta(0)$ is $discontinuous$.[6]

So for $n \in N$ there is a $sequence$ like $\{x_m^n\}_m \in X$ such that $x_m^n\overrightarrow{m \to \infty}\,y_n$.

And $a_nb_n.\theta_m^n\overrightarrow{m \to \infty}\,y_n$ but $y_n \neq 0$ because for every $q_{r_n}(b_n.x_m^n)\overrightarrow{m \to \infty}\,0$, so from order to Next:[7]

$\|a_nb_n.\theta(x_m^n)\| > np_{k_n}(a_n)q_{r_n}(b_n.x_m^n)$

$So\ there\ is\ a\ sequence\ like\ \{x_n\} \subset X,\ such\ that$:

$\|a_nb_n.\theta(x^{\,n})\| > np_{k_n}(a_n)q_{r_n}(b_n.x_n)$

If $q_{r_n}(b_n.x_n)$, $\{\varepsilon_n\} > 0$ exists, so that $\|a_nb_n.\theta(x^{\,n})\| > \varepsilon_n$ Now ⟦ $\lambda_n \in N$ chose so that in relation to:[8]

$$\delta_n = \varepsilon_n - \frac{1}{\lambda_n}\left(1 + \frac{np_{k_n}(a_n)}{1 + \|a_nb_n.\theta(s_n)\|}\right) > 0,$$

replace x_n with $z_n = \varepsilon_n + \frac{s_n}{\lambda_n(1+\|a_nb_n.\theta(s_n)\|)} > 0$.

$$\frac{1}{\lambda_n(1 + \|a_nb_n.\theta(s_n)\|)} - q_{r_n}(b_n.x_n) \leq q_{r_n}(b_n.z_n) \leq \frac{1}{\lambda_n(1 + \|a_nb_n.\theta(s_n)\|)} + q_{r_n}(b_n.x_n).$$

That is, $q_{r_n}(b_n.z_n) = \frac{1}{\lambda_n(1+\|a_nb_n.\theta(s_n)\|)} \neq 0$. On the other

$$np_{k_n}(a_n)q_{r_n}(b_n.z_n) = \frac{np_{k_n}(a_n)}{\lambda_n(1 + \|a_nb_n.\theta(s_n)\|)} = -\delta_n + \varepsilon_n - \frac{1}{\lambda_n}$$

Therefore,

$$\|a_nb_n.\theta(z_n)\| \geq \|a_nb_n.\theta(x_n)\| - \frac{\|a_nb_n.\theta(s_n)\|}{\lambda_n(1 + \|a_nb_n.\theta(s_n)\|)} \geq \varepsilon_n - \frac{1}{\lambda_n} = np_{k_n}(a_n)q_{r_n}(b_n.z_n).$$

there is a $sequence$ like $\{x_n\} \subseteq X$, for every $n \in N$, $\|a_nb_n.\theta(x_n)\| > np_{k_n}(a_n)q_{r_n}(b_n.x_n)$ Now applying the first part for the sequences $\{a_n\}$ and $\{b_n.x_n\}$instead of x_n we can conclude that $C > 0$exists, so that

$\|a_n b_n.\theta(x_n)\| \leq Cpk_m$

for every $n \in N$, the relation $n < C <$ is $established$, which is a $contradiction$.[9]

the posterior $assumption$ is invalid and the $verdict$ is correct, that is, the $linear\ operator\ a_n b_n.\theta(0)$ is $continuous$ from one order to the next. Now we will prove the second part of (b). Because the linear operator:[10]

$$\frac{a_n b_n.\theta}{p_{k_n}(a_n)q_{r_n}(b_n)}$$

$Is\ continuous$, so it takes bounded sets to bounded sets.[11]

let $B(0,1)$ $is\ the\ unit\ open\ sphere\ in\ Y$, because this $operator$ is $continuous$ and the $inverse\ image$ of $open\ sets$ under this operator is an open set, so there exists k,

$$\frac{a_n b_n.\theta(V_k)}{p_{k_n}(a_n)p_{r_n}(b_n)} \subseteq B(0,1),$$

Where $V_k = \left\{x \in X: q_k(k)(x) < \frac{1}{k}\right\}$ since $\{V_k\}$ is a basis for topology on X and E is a bounded set, then there exists one $0 < M$ such that $E \subseteq V_k$. As a result, for all $\| a_n b_n.\theta(x)\| \leq Mp_{k_n}(a_n)p_{r_n}(b_n)$, the rule is proved.[12]

$Theorem$ 2. $(A,\{p_n\})$ is a $Freshe$ algebra, B is a Banach algebra and A → B: is a $homomorphism$. Moreover, $\{p_n\}$ is a $sequence$ in A, such that for every subsequence $a_n a_m = 0$، $n \neq m$, there exists n≠m, so that $p_{k_n}(a_n) \neq 0$ (According to point 1 (a) there is a sub-dial.[13]

(a) If $\{b_n\}$ is a sequence of elements of A, such that for $every$ $a_n b_m = 0$، $\neq m$ and also there exists a subsequence $\{p_{r_n}\}$, such that for every $p_{k_n} b_n \neq 0$ ،n, (there is such a subsequence according to point 1.1.4 (A)), then there is a constant like C>0, so that:

$\|\theta(a_n b_n)\| \leq Cp_{k_n}(a_n)p_{r_n}(b_n)$

(5)

(b) If $\{b_n\}$ is a sequence of elements of A, such that for every $a_n b_m = 0$، there exists $n \neq m$, and also a subsequence $\{p_{r_n}\}$, such that $b_n A \nsubseteq Kerp_{r_n}$, then the linear operator $T_n : A \to B$ with the rule $T_n x = \theta(a_n b_n x)$ is continuous from one order to the next. Also, for every bounded subset such as $E \subseteq X$ there is a constant $0 < M$, so that the following relation holds for every $x \in E$ from one order to the next:

$\theta(a_n b_n x) \leq Mp_{k_n} a_n p_{r_n}(b_n)$

(6)

Argument (A) According to the $assumptions$ of the case, without entering into the whole problem of the problem, it can be assumed:

$p_{k_n}(a_n) = p_{r_n}(b_n) = 1$

$To\ prove\ the\ theorem$, the sentence is not true, that is, there is no Ca that applies to relation (5). As a result, it can be said that there is a $mapping$ like T: N × N → N with the rule $T\left(\left(u_{i,j}\right)\right) = n(i,j)$, so that this mapping is ascending on both components) To obtain T, we use induction (and the fo

$\left\|\theta_{(u_{i,j})}\upsilon_{(i,j)}\right\| \geq 4^{i+j}$. (7)

That $u_{i,j} = a_{n(i,j)}$ and $\upsilon_{(i,j)} = b_{n(i,j)}$. For eachf_i، $i \in N$ as $f_i = \sum_{k=1}^{\infty} 2^{-k} u_{(i,k)}$.

it can be said that $for\ every\ i \in N$ the series is $Homomorphisms$ in A , for all i, choose $j(i)$ so that $j(i) > i$ and $\|\theta f_j\| \leq 2^{\ j(i)}$, define S as follows:

$$S = \sum_{k=1}^{\infty} 2^{-k} u_{(i,k)}$$

As in the discussion of the previous theorem, it can be said that this series is also $Homomorphisms$ in A. On the other hand, for ?,

$f_i S = 2^{-i-j(i)} u_{(i,j(i))}\upsilon_{(i,j(i))}$.

Because θ is $Homomorphisms$, according to (7),

$\|\theta(f_i S)\| = \left\|2^{-i-j}\theta\left(u_{(i,j(i))}\upsilon_{(i,j(i))}\right)\right\| \geq 2^{i+j(i)}$.

(8)

On the other hand, according to the definition of soft operator, we have:

$\|\theta(f_i S)\| \leq \|\theta(S)\|\|\theta(f_i)\| \leq 2^{j(i)}\|\theta(S)\|$ ،i

So with the help of relation (8-4) and the above relation for each $2^i \leq \|\theta(S)\|$ which is a $contradiction$. Therefore, the postulate of $Khalaf\ is\ invalid$, and as a result, the ruling is correct.[14]

(b) $According$ to the $assumptions$ of the $theorem$, the sequence $\{S_n\}$ in A can be chosen such that$p_{rn}(b_n s_n) = 1$. To prove it by the posterior proof method, let us assume that T_n is discontinuous for an infinite number of n ∈ N. Like the argument of the previous theorem, we can say: there is a sequence like $\{x_n\} \subseteq A$ such that for every n ∈ N, $\|T_n(x_n)\| = \|\theta(a_n b_n x_n)\| > np_{k_n}(a_n)p_{r_n}$ and first for the sequences $\{a_n\}$ and $\{b_n x_n\}$ instead of $\{x_n\}$ we can conclude that $C > 0$exists, so that;

Therefore, for every $n \in N$, the relation $n > C$ is established, which is a contradiction. Therefore, the postulate of Khalaf is invalid and the ruling is correct.[15]

To prove the second part (b), as a linear operator,

$$\frac{T_n}{p_{k_n}(a_n)p_{\ rn}(b_n)}$$

Is continuous, so this operator takes bounded sets to bounded sets. Similar to the proof of the previous theorem, it can be said that there exists an $M < 0$, so that for each $x \in E$ from one order onwards, we have $\|T_n(x)\| \leq Mp_{k_n}(a_n)p_{r_n}(b_n)$, and therefore the verdict is confirmed.[16]

5-CONCLUSION

In this paper, we explore the properties of commutative regular Fréchet algebras and the continuity of

homomorphisms under certain conditions. Specifically, we demonstrate that if A is a commutative regular Fréchet algebra and $\pi_m^{-1}(RadA_m) \subseteq kerp_m$, where A_m is the completion of $A/kerp_m$ with respect to the norm $p_m^{\prime}(x + kerp_m) = p_m(x)$ ($x \in A$), and $\pi_m: A \rightarrow A_m$ is the natural projection (Hörmander, 1966), then $A/kerp_m$ is a Fréchet Q-algebra.
Furthermore, we establish that if $(A, \{p_r\})$ is a commutative regular Fréchet algebra satisfying $\pi_r^{-1}(RadA_r) \subseteq kerp_r$ for all sufficiently large $r \in N$, and $(B, \{q_r\})$ is a commutative semisimple Fréchet algebra, then any homomorphism $\tau: A \rightarrow B$ such that $\tau(kerp_r) \subseteq kerq_r$ for all sufficiently large $r \in N$ is continuous (Malliavin, 1995). Moreover, we investigate the automatic continuity of A-module homomorphisms from a Fréchet A-module into a Banach A-module, where A is a unital Fréchet algebra (Waelbroeck, 1971). Finally, we demonstrate that if A is a unital Fréchet algebra with a bounded approximate identity and B is a Banach algebra, then every homomorphism $\theta: A \rightarrow B$ is automatically continuous (Mortini, R., & Rupp, R., 2016).
The results presented in this paper contribute to a deeper comprehension of the properties and behavior of Fréchet algebras, which have applications in various areas of mathematics, including functional analysis, operator theory, and partial differential equations (Taylor, 1958; Hörmander, 1966; Malliavin, 1995; Waelbroeck, 1971; Brudnyi, 2012). Still, it is important to note that our findings are limited to the specific conditions and assumptions outlined in the paper.
Future research could explore the extension of these results to non-commutative Fréchet algebras or investigate the continuity of homomorphisms under different algebraic structures or topological conditions. In addition, studying the connections between Fréchet algebras and other areas of mathematics, such as representation theory or algebraic geometry, could lead to new insights and applications.
Furthermore, the development of computational techniques and algorithms for working with Fréchet algebras could facilitate their practical implementation and enable the exploration of more complex systems and models. Lastly, the application of these results to specific problems in areas like quantum mechanics, signal processing, or control theory could provide valuable insights and potential solutions.
Overall, while this paper makes significant contributions to the understanding of Fréchet algebras, there remains ample opportunity for further exploration and advancements in this field, both theoretically and in terms of practical applications.

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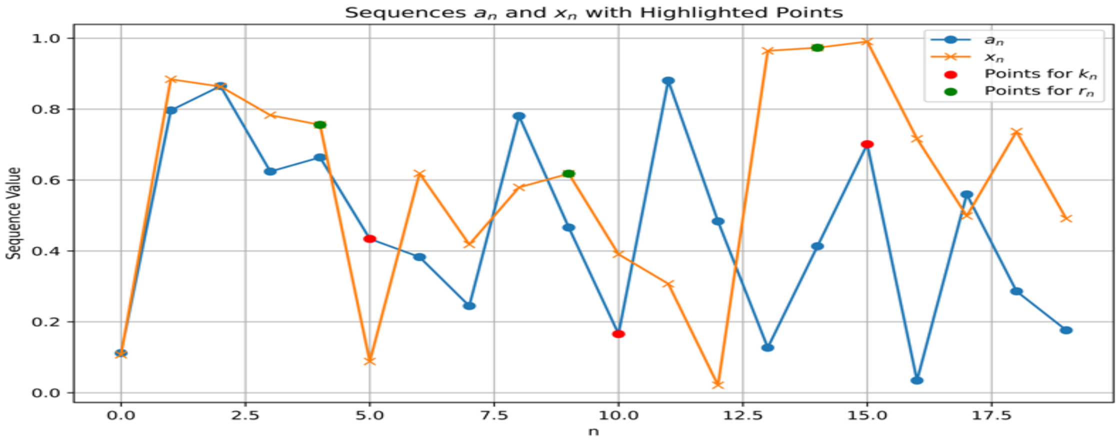
Arabic Abstract

تتناول هذه الورقة البحثية تحويلات الحلقات الجبرية ومفاهيم ذات صلة في نظرية الحلقات. نقدم التعريفات الأساسية بما في ذلك تحويلات الحلقات، الايزومورفيزمات، والأوتومورفيزمات. يتم عرض خصائص التحويلات الجبرية بين الجبر المركب، مع التركيز على الخصائص الضربية والاستقرار. ثم ندرس التحويلات على الجبر فريشيه، مشتقين متباينة تحد من قيمة تحويلات وحدات A حيث A هي جبر فريشيه ذو وحدة. يتم تحليل استمرارية ومحدودية قيمة التحويلات تحت ظروف مختلفة. يتم إنشاء متباينات إضافية لقيمة التحويلات من جبر فريشيه إلى جبر باناخ. يتم إثبات الاستمرارية الأوتوماتيكية للتحويلات من جبر فريشيه مع هويات تقريبية محدودة إلى جبر باناخ. تختتم الورقة بتلخيص النتائج الرئيسية حول استمرارية ومحدودية قيم التحويلات بين الهياكل الجبرية في التحليل الوظيفي. يزيد التطوير النظري من فهم الحفاظ على البنية للحلقات والجبر المجهز ببنى الفضاء الشعاعي التوبولوجي.

Appendix 1

The visual representation of the sequences $\{a_n\}$ and $\{X_n\}$ with the highlighted points corresponding to K_n and T_n:



Appendix 2

The 3D plot of the mapping showing the ascending property T